Lecture 19: Aharonov-Bohm Effect and Magnetic Monopole, Friday, Oct. 21

For a long time, physicists believed that the vector potential $\vec{A}$ is a mathematical object which is introduced just to reproduce a magnetic field. However, it turns out that $\vec{A}$ does have a physical effect even where the magnetic field vanishes.

Consider a charged particle moving in a cylinder with z restricted from 0 to L, and the radius restricted from a to b. The wave function has to vanish on the inner and outer walls and ends of the cylinder. The free Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]+\frac{p_z^2}{2m}\right]\psi = E\psi \ . \tag{114}$$

This equation can be solved using the method of separating variables, $\psi(\rho,\phi,z) = f(\rho)g(\phi)h(z)$.

The equation in the z direction is that of a free particle in one dimension, and can easily be solved to

$$h(z) = A\sin kz + B\cos kz \ , \tag{115}$$

where A/B and k can be determined by the boundary conditions on the lids: $B = 0$, $k = l\pi/L$. The associated quantized energy is $(\hbar^2/2m)(l\pi/L)^2$, with $l = 1, 2, ...$ The equation for $g(\phi)$ can also be solved to yield

$$g(\phi) = e^{\pm im\phi} \ , \tag{116}$$

where $m = 0, \pm1, \pm2, ...$ The radial equation is related to the Bessel equation

$$\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}+\left(1-\frac{m^2}{x^2}\right)f = 0 \ , \tag{117}$$

which has two independent solutions $J_m(x)$, *Bessel function* which is regular at the origin, and $N_m(x)$, *Neumann function* which is singular at the origin. The most general solution is a superpositoin of the two. The energy eigenvalues are

$$E_{lmn} = \left(\frac{\hbar^2}{2m}\right)\left[k_{mn}^2+\left(\frac{l\pi}{L}\right)^2\right] \ , \tag{118}$$

where k_{mn} is the n-th root of the transcendental equation

$$J_m(kb)N_m(ka) - N_m(kb)J_m(ka) = 0 \ . \tag{119}$$

[For further information about the Bessel function, see J. D. Jackson, Classical Electrodynamics, Page 104-105, second edition, sorry I don't h ave a newer edition.]

Now suppose there is a uniform magnetic field confined to $\rho < a$, the vector potential outside the $\rho = a$ can be taken as

$$\vec{A} = \frac{Ba^2}{2\rho}\hat{\phi} \ , \tag{120}$$

which cannot be zero because

$$\int \vec{B} \cdot d\vec{S} = \int \vec{A} \cdot d\vec{\ell} \neq 0 \ . \tag{121}$$

One can verify that this vector potential does give a zero magnetic field. One might wonder, however, since $\vec{B} = 0$ everywhere in the region of interest, why we can't make a gauge transformation to render $\vec{A} = 0$ in the region. The kind of gauge transformation that does this is not be a singularity-free function, therefore is not allowed. The fundamental reason for this is that the space we consider is not simply-connected, or is *topologically nontrivial.* A closed-path integral of the vector potential is in fact a gauge-invariant quantity!

In this case, the partial derivative in the Hamiltonian is changed as follows,

$$\frac{\partial}{\partial \phi} \rightarrow \frac{\partial}{\partial \phi} - \left(\frac{ie}{\hbar c}\right)\frac{Ba^2}{2} \ . \tag{122}$$

Therefore, we need to shift the m by

$$m \rightarrow m - \left(\frac{e}{\hbar c}\right)\frac{Ba^2}{2} \ . \tag{123}$$

Therefore the energy-eigenvalues do change! This is a form of the Aharonov-Bohm effect. The moral of the above discussion is that the vector potential, or at least the closed-path integral of it, is physical, not just a mathematical convenience. On the other hand, if the change in m is a multiple of an integer, the wave functions and energy levels remain the same. In this case, we have the magnetic flux quantized,

$$\pi a^2 B = \frac{2\pi\hbar c}{e} n \ , \tag{124}$$

where n is integer $0, \pm 1, \pm 2.....$ In a superconductor ring, the flux is indeed quantized. Therefore the Aharonov-Bohm effects disappear if the magnetic field is produced through a super-conducting ring.

Aharonov-Bohm effects

The original Aharonov-Bohm effect refers to the interference between particles travelling different paths through a region with non-vanishing magnetic fields. Consider a particle source emitting particles travelling by an impenetrable cylinder which contains non-vanishing magnetic fields. The particles are brought together to interfere afterward. There are no magnetic fields on the paths that the particles traverse through.

The classical action is

$$L = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{e}{c}\frac{dx}{dt} \cdot A \ , \tag{125}$$

which depends on $\vec{A}$. According to the Feynman psath integral formalism the probability amplitude for particles travelling above the cylinder and below the cylinder has a phase difference,

$$\frac{e}{\hbar c}\int \vec{A} \cdot d\vec{\ell}\Big|_{\text{above}} - \frac{e}{\hbar c}\int \vec{A} \cdot d\vec{\ell}\Big|_{\text{below}} = \frac{e}{\hbar c}\oint \vec{A} \cdot d\vec{\ell} = \frac{e}{\hbar c}\Phi_B \tag{126}$$

where Φ_B is the magnetic flux. Therefore, the interference pattern depends on the total flux! When the flux is the multiple of

$$\frac{2\pi\hbar c}{e} = 4.135 \times 10^{-7} \quad \text{Gauss} \cdot \text{cm}^2 \tag{127}$$

the interference is constructive. This has been observed experimentally [see A. Tonomura et al, Phys. Rev. 48, 1442 (1982).]

Magnetic monopole and charge quantization

A magnetic monopole is an object that produces the following magnetic field,

$$\vec{B} = \frac{g}{r^2}\vec{r} \ , \tag{128}$$

It turned out that one cannot reproduce this magnetic field with a single vector potential that is regular everywhere. It can be, however, reproduced with the following two potentials

$$\begin{aligned} \vec{A} &= \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for} \;\; \theta < \pi - \epsilon \\ \vec{A} &= -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\phi} \quad \text{for} \;\; \theta > \epsilon \end{aligned} \tag{129}$$

And in the region that they overlap, there are related by a gauge transformation with $\Lambda = -2g\phi$. The wave functions in these two gauges are related by

$$\psi' = \exp\left(\frac{-2ieg\phi}{\hbar c}\right)\psi \ , \tag{130}$$

Since a wave function must be single-valued, it must return to the original value after a rotation of 2π. This means that following must be true,

$$\frac{2eg}{\hbar c} = \pm n \ . \tag{131}$$

where n is an integer. Therefore the magnetic charge must be quantized or else the quantum mechanics is inconsistent! On the other hand, if a magnetic monopole exists, we can argue in the reverse direction that the electric charge must be quantized.