Homework 11: Due November 24

1.

Consider the operator $\hat{\Theta} = \hat{J}_x^2 + \hat{J}_y^2 - 2\hat{J}_z^2$.

a. Show that $\hat{\Theta}$ can be written as the *q=0* component of a rank two spherical tensor, *i.e.* $\hat{\Theta} = \hat{T}_0^2$ .Find the general vector $\hat{T}_q^2$

b. Explicitly calculate the matrix element $\langle jm' | \hat{\Theta} | jm \rangle$ using general properties of angular momentum

c. Use the Wigner-Eckart theorem to show that $\dfrac{\langle jm_1 | \hat{\Theta} | jm_1 \rangle}{\langle jm_2 | \hat{\Theta} | jm_2 \rangle} = \dfrac{\langle j\, 2\, m_1\, 0 | jm_1 \rangle}{\langle j\, 2\, m_2\, 0 | jm_2 \rangle}$

d. Verify that formulae in c. satisfy the ratio's in b. for the case if *j*=2 and all *m*. It is sufficient to check the ratio of *m*=2 to *m*=1; *m*=1 to *m*=0; *m*=0 to *m*=-1; *m*=-1 to *m*=-2.

2.

The states of the three dimensional isotropic harmonic oscillator is often labeled by the Cartesian labels $|n_x n_y n_z\rangle$ these states are highly degenerate. It is possible to take linear combinations of these to obtain states with good *l* and *m*: $|n\,l\,m\rangle$. The purpose of this problem is to use tensorial properties to construct some of these $|n\,l\,m\rangle$ in terms of the $|n_x n_y n_z\rangle$ basis. We will work using creation operators for the three Cartesian directions

a. Show explicitly that $\hat{V}_\mu^1$ where $\hat{V}_0^1 = \hat{a}_z^+$ and $\hat{V}_{\pm 1}^1 = \mp \dfrac{(\hat{a}_x^+ \pm i\hat{a}_y^+)}{\sqrt{2}}$ is a rank one tensor by verifying it has the correct commutation rules with $\hat{\vec{L}}$.

b. Explicitly construct the tensor operators $\hat{S} = \sum_{\mu'} \langle 1 1 \mu' (-\mu') | 00 \rangle \hat{V}_{\mu'}^1 \hat{V}_{-\mu'}^1$
$\hat{T}_2^2 = \sum_{\mu'} \langle 1 1 \mu' (2-\mu') | 22 \rangle \hat{V}_{\mu'}^1 \hat{V}_{2-\mu'}^1$ $\hat{T}_1^2 = \sum_{\mu'} \langle 1 1 \mu' (1-\mu') | 21 \rangle \hat{V}_{\mu'}^1 \hat{V}_{1-\mu'}^1$
$\hat{T}_0^2 = \sum_{\mu'} \langle 1 1 \mu' (-\mu') | 20 \rangle \hat{V}_{\mu'}^1 \hat{V}_{-\mu'}^1$ in terms of $\hat{a}_x^+, \hat{a}_y^+, \hat{a}_z^+$. By construction these are sphereical tensors.

c. Operate $\hat{V}_\mu^1$, $\hat{S}$ and $\hat{T}_\mu^2$ on the ground state and exploit the Wigner –Eckart theorem to find the states $|n\,l\,m\rangle = |111\rangle, |110\rangle, |11-1\rangle, |200\rangle, |222\rangle, |221\rangle, |220\rangle$ in terms of the states $|n_x n_y n_z\rangle$.

3. This problem explores some properties of $\hat{\pi}$, the parity operator
   a. Show that $\hat{\pi}$ is both unitary and hermitian.
   b. The fact that $\hat{\pi}$ is unitary allows one to write it as $\hat{\pi} = \exp(i\pi \hat{G}/2)$ where $\hat{G}$ is Hermitian (and π on the righthand side is the number 3.14… and not the operator). Show that $\hat{G} = \hat{\pi} - \hat{1}$.
   c. Construct the operator $\hat{U}(\theta) = \exp(i\hat{G}\theta/2)$; by construction $\hat{U}(\pi) = \hat{\pi}$
   d. Show that $\hat{U}(\theta)$ is periodic: $\hat{U}(\theta + 2\pi) = \hat{U}(\theta)$

4. In class it was stated that $\hat{\pi}$ was a discrete symmetry unobtainable by making a continuous rotation. In problem 3, however you have demonstrated that $\hat{\pi}$ can be obtained continuously from $\hat{U}(\theta)$. This problem explores the action of $\hat{U}(\theta)$
   a. Find an expression $\hat{\vec{x}}' = \hat{U}^+(\theta)\hat{\vec{x}}\hat{U}(\theta)$; this tell one the effect of the continuous transformation on $\hat{\vec{x}}$.
   b. Does the result in a. have any analog in classical physics? explain.

Sakurai Chapter 3, 28, 29.