# Homework 10: Due November 17

1. Clebsch-Gordan coefficients can be expressed in the form $\langle j_1 j_2 m_1 m_2 \mid jm\rangle$. Here I want you to compute from first principles all nonvanishing Clebsch-Gordan coefficients obtained from combining to angular momenta with j=1. That is you should calculate

$$\langle 1111|22\rangle\ \langle 1101|21\rangle\ \langle 1110|21\rangle\ \langle 11{-1}1|20\rangle\ \langle 1100|20\rangle\ \langle 111{-1}|20\rangle\ \langle 11{-1}0|2{-1}\rangle\ \langle 110{-1}|2{-1}\rangle\ \langle 11{-1}{-1}|2{-2}\rangle$$

$$\langle 1101|11\rangle\ \langle 1110|11\rangle\ \langle 11{-1}1|10\rangle\ \langle 1100|10\rangle\ \langle 111{-1}|10\rangle\ \langle 11{-1}0|1{-1}\rangle\ \langle 110{-1}|1{-1}\rangle$$

$$\langle 11{-1}1|00\rangle\ \langle 1100|00\rangle\ \langle 111{-1}|00\rangle$$

Do this by noting that the state $|22\rangle = |1111\rangle$ and using the lowering operator and orthogonality. Alternatively you may use the recursion relation in the book. Compare your results with a C-G table.

2. Consider two Cartesian vectors $\vec{A}$ and $\vec{B}$. There are several ways to take products of the two to produce Cartesian tensors of various ranks: one can produce a scalar, namely the dot product $\vec{A}\cdot\vec{B}$ the cross product $\vec{A}\times\vec{B}$ and traceless symmetric tensor $\bar{C}$ whose components are given by $C_{ij} = \frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}\delta_{ij}\vec{A}\cdot\vec{B}$. Note that the components of these three structures are linearly independent: there are nine possible combination of $A_i B_j$ and there is one dot product, three components of the cross product and 5 independent components of C (once the fact that it is traceless and symmetric is included). This same information can be expressed in terms of three spherical tensor: an l=0, and l=1 and l=2. Derive the forms for these in terms of the Cartesian components $A_i, B_j$ from first principles. In effect this is like deriving 3.10.16 but for a more general case Express Do not start from 3.10.16

3. Consider a system of three distinguishable spin ½ particles whose positional degrees of freedom are irrelevant (held fixed dynamically). Clearly there are a total of 8 states: $|\uparrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\rangle$ etc. Obviously, the total spuin must by either 3/2 or 1/2. Moreover there must be two different linear independent ways to make the total spin ½ (otherwise one would only have 6 states:4 from the spin 3/2 and 2 from the ½). One way to make these states is to first couple spins 1 & 2 together to get $s_{12}$ which has a magnitude of either 1 or 0 and then coupe this to the third spin to get a total spin. Thus the states can be labeled by $|s_{12}\ s_{total}\ m\rangle$. Thus for example the $|1\frac{3}{2}\frac{3}{2}\rangle$ in the $|s_{12}\ s_{total}\ m\rangle$ basis is equal to $|\uparrow\uparrow\uparrow\rangle$ in the $|m_1 m_2 m_3\rangle$ basis. Find all of the states in the $|s_{12}\ s_{total}\ m\rangle$ basis as a superposition of states in the $|m_1 m_2 m_3\rangle$.

4. Consider the system described in problem 3. This problem concerns operators for that system. One can construct a rank three spherical tensor which acts on these states. As it happens in this small basis of states there is only one such operator (up to an overall constant); let us label the operator $\hat{T}^{(3)}_{\mu}$ One way to construct this operator is in terms of products of the Pauli spin operators acting on each of the three separate spins $\hat{\vec{\sigma}}_1, \hat{\vec{\sigma}}_2$ and $\hat{\vec{\sigma}}_3$. It is easier to work with these Pauli operators in a spherical basis, *eg.* $\hat{\sigma}_{\mu 1}$, $\hat{\sigma}_{\mu 2}$ and $\hat{\sigma}_{\mu 3}$ where $\hat{\sigma}_{01} = \hat{\sigma}_{z1}$, $\hat{\sigma}_{\pm 11} = \mp\frac{1}{\sqrt{2}}(\hat{\sigma}_{x1} \pm i\hat{\sigma}_{y1})$ and analogously for the

operators acting on particle 2 and particle 3. The purpose of this problem is to express $\hat{T}_{\mu}^{(3)}$ as linear combinations of operators of the form $\hat{\sigma}_{\mu_1\,1}\hat{\sigma}_{\mu_2\,2}\hat{\sigma}_{\mu_3\,3}$.

a. As a first step show that $\hat{T}_{+3}^{(3)} = A\hat{\sigma}_{+1\,1}\hat{\sigma}_{+1\,2}\hat{\sigma}_{+1\,3}$ where A is a constant.

b. It should be clear that $\hat{T}_{+2}^{(3)} = B\hat{\sigma}_{0\,1}\hat{\sigma}_{+1\,2}\hat{\sigma}_{+1\,3} + C\hat{\sigma}_{+1\,1}\hat{\sigma}_{0\,2}\hat{\sigma}_{+1\,3} + D\hat{\sigma}_{+1\,1}\hat{\sigma}_{+1\,2}\hat{\sigma}_{03}$. Find B,C, and D. You can do this by using $[\hat{J}_{-},\hat{T}_{+3}^{(3)}]$ and the fact that $\hat{J}_{-} = \hat{J}_{1-} + \hat{J}_{2-} + \hat{J}_{3-}$.

c. Generate $\hat{T}_{+1}^{(3)}$ and $\hat{T}_{0}^{(3)}$ in a similar fashion. (You do not need to compute $\hat{T}_{-1}^{(3)}$ $\hat{T}_{-2}^{(3)}$ or $\hat{T}_{-3}^{(3)}$)

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