Homework 6: Due October 20

Please use units with $\hbar = 1$

1. Verify that the causal propagator for the one dimensional Harmonic oscillator is given by

$$K^C(x',t';x,0) = \langle x'|\hat{U}(t')|x\rangle \theta(t') =$$
$$\sqrt{\frac{m\omega}{2\pi i \sin(\omega t')}} \exp\left(\frac{i m\omega\left((x^2+x'^2)\cos(\omega t') - 2x\,x'\right)}{2\sin(\omega t')}\right)\theta(t') \quad .$$

That is you should show that
$\left(\frac{-1}{2m}\frac{\partial^2}{\partial x'^2} + \frac{m\omega^2 x'^2}{2} - i\frac{\partial}{\partial t'}\right)K^C(x',t';x,0) = -i\ \delta(x'-x)\delta(t')$. The book has various suggestions as to how you might prove this.

2. In class it was argued that the time-evolution operator at imaginary time is related to the partition function. This has important implications for the propagator In particular one expects that $\int dx\ K^c(x,-i\beta;x,0) = \sum_n e^{-\beta E_n}$ where $Z(\beta) \equiv \sum_n \exp(-\beta E_n)$ is the partition function in statistical mechanics (for which $\beta = \frac{1}{kT}$ where *k* is Boltzmann's constant) and $K^c(x,-i\beta\hbar\backslash;x,0)$ is understood in the sense of an analytic continuation from the real function. In this problem, we focus on the case of the one dimensional harmonic oscillator.

   a. By directly summing using the harmonic oscillator energy levels $E_n = \omega(n+\frac{1}{2})$ show that $\sum_n \exp(-\beta E_n) = \frac{1}{\exp(\beta\omega/2) - \exp(-\beta\omega/2)}$.

   b. Starting from the propagator for the harmonic oscillator derived in last week's homework
   $$K^C(x',t';x,0) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega t')}} \exp\left(\frac{i m\omega\left((x^2+x'^2)\cos(\omega t') - 2x\,x'\right)}{2\sin(\omega t')}\right) \quad .$$
   Evaluate $\int dx\ K^c(x,-i\beta;x,0)$ and show that you reproduce the partition function from part a.

3. Equation 2.5.16 gives the propagator for a free particle (i.e. one for which the potential is zero.) Use this propagator to solve the following problem: Suppose that one has a system which is in the ground state of a harmonic oscillator for a particle of mass *m* and frequency ω. Suppose, further at t=0 potential is instantly tuned off leaving the system as a free particle..

a. Use the propagator in 2.5.16 to find the wavefunction for all times t>0.
b. Using this wavefunction evaluate the following **as a function of tim**e:
   i. $\langle x \rangle$
   ii. $\langle x^2 \rangle$
   iii. $\langle p \rangle$
   iv. $\langle p^2 \rangle$
   v. $\Delta x \Delta p$

This problem may appear to be artificial problems worthy only of a quantum mechanics class. However, given recent advances in laser trapping of atoms, one can realize an approximation of this problem in the lab by first trapping an atom and then rapidly turning of the trap..

4. This problem is similar to problem three, with the system in which is in the ground state of a harmonic oscillator for a particle of mass *m* and frequency ω. Suppose, in this problem at t=0, instead of the potential being instantly tuned off, the strength of the potential instantly doubles, leaving the system as a harmonic oscillator with frequency $\sqrt{2}\,\omega$.
   a. Use the form of the propagator derived in problem 1 (using the appropriate frequency) to find the wavefunction for all times t>0.
   b. Using this wavefunction evaluate the following **as a function of tim**e:
      i. $\langle x \rangle$
      ii. $\langle x^2 \rangle$
      iii. $\langle p \rangle$
      iv. $\langle p^2 \rangle$
      v. $\Delta x \Delta p$

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