622 Problem Set 4—Due Monday 10/6/08

Units with $\hbar = 1$ are used throughout

1. Consider a general two level quantum system---that is a system in which only two quantum levels are accessible. Two orthonomal states corresponding to energy eigenstates are labeled $|a\rangle$ and $|b\rangle$. This could be spin states but as a matter of principle it need not be---it could be two low lying levels in a well where all other levels are energetically inacessible. The Hamiltonian for the system is given for the system is $\hat{H} = E_a|a\rangle\langle a| + E_b|b\rangle\langle b|$ with. $E_a \neq E_b$.

Physical observables in quantum mechanics are associated with Hermitian operators. For a two dimensional system, the most general Hermitian operator can be specified by four real numbers . We will denote them *f, g, h* and ϕ and denote the operator $\hat{\Theta}_{f,g,h,\phi}$. It is given by

$$\hat{\Theta}_{f,g,h,\phi} = f|a\rangle\langle a| + g|b\rangle\langle b| + h\,e^{i\phi}|a\rangle\langle b| + h\,e^{-i\phi}|b\rangle\langle a| .$$

a. Verify that $\hat{\Theta}_{f,g,h,\phi}$ is Hermitian.

b. Show that the Heisenberg operator given by $\hat{\Theta}^{H}_{f,g,h,\phi}(t) = \hat{U}^{+}(t)\,\hat{\Theta}_{f,g,h,\phi}\,\hat{U}(t)$ is given by $\hat{\Theta}^{H}_{f,g,h,\phi}(t) = f|a\rangle\langle a| + g|b\rangle\langle b| + h\,e^{i(\phi+\omega t)}|a\rangle\langle b| + h\,e^{-i(\phi+\omega t)}|b\rangle\langle a|$ with $\omega = E_a - E_b$.

c. Suppose at time t=0 the state is given by $|\psi(0)\rangle = c_a|a\rangle + c_b|b\rangle$ where the *c* coefficients are generally complex. Use the result in b. to find an expression for the expectation value of the (Schrodinger picture) operator $\hat{\Theta}_{f,g,h,\phi}$ as a function of time. That is find $\langle\psi(t)|\hat{\Theta}_{f,g,h,\phi}|\psi(t)\rangle$.

2. Consider a one-dimension system with a Hamiltonian given by $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. This problem concerns a useful representation of the time evolution operator. It turns out to be convenient to use the *causal* time evolution operator. This operator satisfies

$$\left(\hat{H} - i\frac{\partial}{\partial t}\right)\hat{U}^{c}(t,t_0) = -i\hat{1}\delta(t-t_0)$$

subject to the boundary condition that $\hat{U}^{c}(t,t_0) = 0$ for $t < t_0$ (where $\hat{1}$ indicates the identity operator).

a. Show that in general $\hat{U}^c(t,t_0)$ is given by

$\hat{U}^c(t,t_0) = \exp\left(-i\hat{H}(t-t_0)\right)\theta(t-t')$ where $\theta(t-t_0) = \begin{cases} 0 & \text{for } t < t_0 \\ 1 & \text{for } t > t_0 \end{cases}$. That is $\hat{U}^c(t,t_0)$ is the usual time evolution operator for $t > t_0$ and zero otherwise.

Thus from a), we see that the causal time evolution operator for a free particle (*i.e.* one for which the potential is zero) is given by: $\hat{U}_0^c(t,t_0) = \exp\left(\frac{-i\,\hat{p}^2(t-t_0)}{2m}\right)\theta(t-t')$ and satisfies

$$\left(\frac{\hat{p}^2}{2m} - i\frac{\partial}{\partial t}\right)\hat{U}_0^c(t,t_0) = -i\hat{1}\delta(t-t_0).$$

b. Show that the causal time evolution operator satisfies the equation

$$\hat{U}^c(t,t_0) = \hat{U}_0^c(t,t_0) - i\int_{t_0}^{\infty} dt_1 \hat{U}^c(t,t_1)V(\hat{x})\hat{U}_0^c(t_1,t_0)$$

where $\hat{U}_0^c(t,t_0)$ is the causal time evolution operator for a free particle given above.

c. While the result in b., seems somewhat formal it, is a useful starting point for a series expansion. Show that if $\hat{U}^c(t,t_0)$ is given by the following series

$$\hat{U}^c(t,t_0) = \hat{U}_0^c(t,t_0) + \hat{U}_1^c(t,t_0) + \hat{U}_2^c(t,t_0) + \hat{U}_3^c(t,t_0) + \ldots.$$

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with $\hat{U}_{n+1}^c(t,t_0) = -i\int_{t_0}^{\infty} dt' \hat{U}_n^c(t,t')V(\hat{x})\hat{U}_0^c(t',t_0)$,

then, the equation in b. is solved.

Note the $\hat{U}_n^c$ can easily be found explicitly by "bootstrapping": from $\hat{U}_0^c$ we can get $\hat{U}_1^c$; then from $\hat{U}_1^c$ we can get $\hat{U}_2^c$ and so forth.

This is called the Born series and is very useful in situations in which the potential is small compared to other energy scales in the problem. Note that each term in the series has one more application of $V(\hat{x})$ compared to the previous one. If the potential is small, then the series should converge rapidly. In practice this allows truncations at relatively low order. The Born series---written in a somewhat different form---plays a key role in scattering theory.

Sakurai---Chapter 2: 11, 15, 16, 19 (This last problem is the basis for Schwinger's elegant formalism for angular momentum).