

622 Problem Set 3—Due Monday 9/29/08

You make use units with $\hbar = 1$ throughout this homework

1. Consider a spin $\frac{1}{2}$ particle in a magnetic field whose magnitude is constant but whose direction rotates in the x-y plane with angular frequency Ω . The Hamiltonian for such a system can be written as $\hat{H} = -\omega(\hat{s}_x \cos(\Omega t) + \hat{s}_y \sin(\Omega t))$ where ω is a constant which depends on the magnetic field and the gyromagnetic ratio. The purpose of this problem is to find the time-evolution operator satisfying $\hat{H}\hat{U} = i\frac{d\hat{U}}{dt}$ with $\hat{U}(0) = \hat{1}$.

- a. As a first step show that $\hat{H} = -\hat{W}^+(\omega\hat{s}_x)\hat{W}$ where \hat{W} is a time dependent unitary transformation given by $\hat{W} = \exp(i\hat{\sigma}_z\Omega t/2)$. A couple of identities you might want to prove first in deriving this are:

- i. $\{\hat{\sigma}_x, \hat{\sigma}_z\} = 0$ (anti-commutator)
- ii. $[\{\hat{\sigma}_z, \hat{\sigma}_x\}] = 2i\hat{\sigma}_y$
- iii. $\hat{\sigma}_z^2 = \hat{1}$

- b. Show that $\hat{U}(t)$ can be written as $\hat{U}(t) = \hat{W}^+(t)\hat{V}(t)$ where $\hat{V}(t)$ satisfies an effective time-evolution equation: $-(\omega\hat{s}_x + \Omega\hat{s}_z)\hat{V} = i\hbar\frac{d\hat{V}}{dt}$.

- c. Show that $\hat{V} = \cos(\omega't/2)\hat{1} + i\sin(\omega't/2)\frac{\omega\hat{\sigma}_x + \Omega\hat{\sigma}_z}{\omega'}$ where $\omega' = \sqrt{\omega^2 + \Omega^2}$.

- d. Use the expressions in b. and c. to find the matrix elements of $\hat{U}(t)$ in the $|+\rangle, |-\rangle$ basis.

2. In principle we know that the time evolution operator in problem 1 can be written as

$$\hat{U}(t) = T\left(\exp\left(-i\int_0^t dt' \hat{H}(t')\right)\right). \text{ Suppose we wish to approximate this by expanding the}$$

$$\text{exponential to second order: } \hat{U}(t) \approx \hat{1} + T\left(-i\int_0^t dt' H(t')\right) + \frac{1}{2}T\left(-i\int_0^t dt' H(t')\right)^2.$$

- a. Explain qualitatively why this should correspond to expanding the exact solution up to second order in a Taylor series in ω .
- b. Compute the Taylor series for the exact solution up to second order in ω . You may find it VERY helpful to do this via Mathematica or other symbolic manipulation program.
- c. Calculate the time ordered products and compare with part b.