Physics 622

Take home Exam----Due 10:00 A.M. Monday, October 27

This exam is open notes and open book. You may also use *Mathematica* or other symbolic manipulation programs. **If you use *Mathematica* or a similar program you must include the output to get credit.** Do not seek outside help. (I trust you.) **To get credit you must show how you obtained your answer from the basic physical and mathematical principles. You may use formulae that we derived in class or in the book as a starting point. Since you have considerable time on this exam, I fully expect your answers to be clear.**
If you have questions you may e-mail me (cohen@physics.umd.edu) or call me at the office (301) 405-6117 or at home (301) 654-7702 (**Before 10:00 p.m.**) **Before contacting me check the course website. If a clarification of a question is needed I will post the corrected version.**

A word of encouragement: while this exam looks long it should not take an overly long time to complete. Part of the length is due to a considerable amount of explanatory information with each problem. Most of these problems are conceptual in nature and do **NOT** require long and ugly calculations. A few of the problems are *very* easy. If you are worried that a particular answer must be wrong because the problem seemed too easy---relax: it probably was that easy! The exam is written in such a way that you can often do a later section of a problem while missing earlier parts.

Use units with $\hbar = 1$

1. This problem concerns a statistical ensemble of spin-1/2 systems. Such a system is naturally characterized by a density operator with the property that the expectation value of any observable A is given by $\langle\langle A \rangle\rangle = tr(\hat{A}\hat{\rho}) = \sum_{i,j} A_{ij}\rho_{ji}$ where $A_{ij} = \langle i|\hat{A}|j\rangle$ in some orthonormal basis. The density operator $\hat{\rho}$ --like any two-dimensional Hermitian matrix--may be parameterized in the following form: $\hat{\rho} = r_0\hat{1} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z$ where r_0, r_x, r_y, r_z are four real constants. (Any two-dimensional Hermitian operator can be put in this form). Since $tr(\hat{\rho}) = 1$ for any density matrix we know that $r_0 = \frac{1}{2}$ regardless of the state of the system .

   a. Show that the system is in a "pure state" (*i.e.,* one described by a single well-defined quantum state) if, and only if, $r_x^2 + r_y^2 + r_z^2 = \frac{1}{4}$. Hint: What are the properties of the density matrix of a pure state?

   b. There is a constraint on $r_x^2 + r_y^2 + r_z^2$ for a mixed state: namely, $r_x^2 + r_y^2 + r_z^2 < \frac{1}{4}$. Derive this constraint.

   c. Several 2×2 matrices are given below. For each one indicate whether it corresponds to the density matrix for a pure state, corresponds to the density matrix of a mixed state, or does not correspond to any density matrix:

$$\text{i).}\ \begin{pmatrix} \frac{1}{2} & \frac{1-i}{2\sqrt{2}} \\ \frac{1+i}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad \text{ii).}\ \begin{pmatrix} \frac{3}{4} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{4} \end{pmatrix} \quad \text{iii).}\ \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}\,i}{4} \\ \frac{\sqrt{7}\,i}{4} & \frac{1}{4} \end{pmatrix} \quad \text{iv).}\ \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{v).}\ \begin{pmatrix} \frac{3}{4} & -\frac{i}{4} \\ \frac{i}{4} & \frac{3}{4} \end{pmatrix}$$

2. In homework 4 you showed---or should have shown---that the causal time evolution operator for a one-dimensional system with a time-independent Hamiltonian given by $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ satisfies the equation

$$\hat{U}^c(t,t_0) = \hat{U}_0^c(t,t_0) - i\int_{t_0}^{\infty} dt_1 \hat{U}^c(t,t_1)V(\hat{x})\hat{U}_0^c(t_1,t_0)$$

where $\hat{U}^c(t,t_0)$ and $\hat{U}_0^c(t,t_0)$ satisfy $\left(\hat{H} - i\frac{\partial}{\partial t}\right)\hat{U}^c(t,t_0) = -i\hat{1}\delta(t-t_0)$ and $\left(\frac{\hat{p}^2}{2m} - i\frac{\partial}{\partial t}\right)\hat{U}_0^c(t,t_0) = -i\hat{1}\delta(t-t_0)$, respectively. This problem explores some consequences of this.

a. One useful result is an integral equation for the causal propagator. Stating with the integral equation for the causal time evolution operator, show that the causal propagator satisfies

$$K^c(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i(t-t_0)}}\exp\left(\frac{im(x-x_0)^2}{2(t-t_0)}\right)\theta(t-t_0)$$
$$-i\int_{t_0}^{t} dt_1 \int_{-\infty}^{\infty} dx_1\, K^c(x,t;x_1,t_1)V(x_1)\sqrt{\frac{m}{2\pi i(t_1-t_0)}}\exp\left(\frac{im(x_1-x_0)^2}{2(t_1-t_0)}\right)\theta(t-t_0).$$

Note: you do NOT need to re-derive the expression for the free propagator that we derived in class.

b. Show that the causal propagator can be written as the following series:

$$K^c(x,t;x_0,t_0) = \sum_{n=0}^{\infty} K_n^c(x,t;x_0,t_0)$$

with $K_0^c(x,t;x_0,t_0) = \sqrt{\frac{m}{2\pi i(t-t_0)}}\exp\left(\frac{im(x-x_0)^2}{2(t-t_0)}\right)\theta(t-t_0)$

and $K_{n+1}^c(x,t;x_0,t_0) = -i\int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dx'\, K_n^c(x,t;x',t')V(x')\sqrt{\frac{m}{2\pi i(t'-t_0)}}\exp\left(\frac{im(x'-x_0)^2}{2(t'-t_0)}\right)\theta(t-t_0)$

c. It is sometimes useful to consider the operator $\hat{G}(E)$ which is defined as $\hat{G}(E) \equiv \int dt \hat{U}^c(t,0)\exp\left(i(E+i\varepsilon)t\right)$ where the limit $\varepsilon \to 0$ is implicitly taken at the end of any calculation. Starting with the integral equation for $\hat{U}^c(t,t_0)$ show that $\hat{G}(E)$ satisfies the $\hat{G}(E) \equiv \hat{G}_0(E) - i\hat{G}(E)V(\hat{x})\hat{G}_0(E)$ where $\hat{G}_0(E) \equiv \int dt \hat{U}_0^c(t,0)\exp\left(i(E+i\varepsilon)t\right)$.

d. Show that $\hat{G}_0(E)$ defined in part c. is given by $\hat{G}_0(E) = i\int dp \frac{|p\rangle\langle p|}{E - p^2/(2m) + i\varepsilon}$.

3. In class we showed that in three-dimensional quantum mechanics describing the motion of a single particle the probability is conserved *locally* as an operator equation in the sense that the continuity equation was

satisfied for Heisenberg operators: $\frac{d\,\hat{\rho}^H(\vec{x})}{dt} = -\nabla\cdot\hat{\vec{J}}^H(\vec{x})$. This problem focuses on another conservation law: momentum. We will focus on a description of a free particle in three dimensions (so $\hat{H} = \frac{\hat{\vec{p}}\cdot\hat{\vec{p}}}{2m}$). Clearly such a system has a conserved momentum ($[\hat{H},\hat{\vec{p}}] = 0$). The purpose of this problem is to describe a *local* conservation law for momentum at the operator level. The form of this equation is $\frac{d\,\hat{\vec{\mathscr{P}}}^H(\vec{x})}{dt} = -\vec{\nabla}\cdot\hat{\vec{\mathscr{T}}}^H(\vec{x})$ where the superscript indicates the Heisenberg picture. Note that $\hat{\vec{\mathscr{P}}}$ ---the momentum density---is a vector and $\hat{\vec{\mathscr{T}}}$ ---the momentum current density (also known as the stress-tensor)---is a matrix. The divergence acting on a matrix yielding a vector may not be familiar to you but it is easy to see its meaning on a component-by-component basis: $\frac{d\,\hat{\mathscr{P}}_j^H(\vec{x})}{dt} = -\sum_k \frac{\partial\hat{\mathscr{T}}_{kj}^H(\vec{x})}{\partial x_k}$. As a first step let us define $\hat{\vec{\mathscr{P}}}(\vec{x}) \equiv \tfrac{1}{2}\left\{\hat{\vec{p}},\hat{\rho}(\vec{x})\right\}$ with $\hat{\rho}(\vec{x}) = \left|\vec{x}\right\rangle\left\langle\vec{x}\right|$ (and, as usual, the braces indicate an anti-commutator).

a. Show that this is a sensible definition for a momentum density in the sense that $\int d^3x\,\hat{\vec{\mathscr{P}}}(\vec{x}) = \hat{\vec{p}}$.

b. Show that if $\hat{\vec{\mathscr{T}}}$ is sensibly defined as $\hat{\mathscr{T}}_{ij}(\vec{x}) \equiv \frac{\{\{\hat{p}_i,\hat{\rho}(\vec{x})\},\hat{p}_j\}}{4m}$; that is, if one adopts this definition then the local conservation of momentum given by the equation $\frac{d\,\hat{\mathscr{P}}_j^H(\vec{x})}{dt} = -\sum_k \frac{\partial\hat{\mathscr{T}}_{kj}^H(\vec{x})}{\partial x_k}$ holds.

c. Show that the expressions for the expectation value of $\hat{\vec{\mathscr{P}}}(\vec{x})$ in a state $\left|\psi(t)\right\rangle$ is given by $\left\langle\vec{\mathscr{P}}(\vec{x})\right\rangle = \tfrac{1}{2}\left(\psi^*(\vec{x},t)\left(-i\vec{\nabla}\psi(\vec{x},t)\right) + \left(-i\vec{\nabla}\psi(\vec{x},t)\right)^*\psi(\vec{x},t)\right)$ where $\psi(\vec{x},t)$ is $\left\langle\vec{x}\middle|\psi(t)\right\rangle$.

d. Find an analogous expression for $\left\langle\mathscr{T}_{ij}(\vec{x})\right\rangle$ (where the operator is defined in part b.)

4. Consider a particle of mass *m* and charge *q* moving in a magnetic field. The Hamiltonian is given by $\hat{H} = \frac{\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{x}})\right)\cdot\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{x}})\right)}{2m}$ where the vector potential satisfies $\vec{\nabla}\times\vec{A}(\vec{x}) = \vec{B}(\vec{x})$ It is useful to define a velocity operator: $\hat{\vec{v}} \equiv -i[\hat{\vec{x}},\hat{H}] = \frac{\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{x}})\right)}{m}$

a. Is the expectation value of $\hat{\vec{v}}$ in a state $\left|\psi\right\rangle$ gauge invariant; that is, does $\left\langle\psi\middle|\hat{\vec{v}}\middle|\psi\right\rangle = \left\langle\psi'\middle|\hat{\vec{v}}'\middle|\psi'\right\rangle$ where $\hat{\vec{v}}' = \left(\hat{\vec{p}} - q\vec{A}'(\hat{\vec{x}})\right)$/m and $\vec{A}'$ and $\left|\psi'\right\rangle$ are related to $\vec{A}$ and $\left|\psi\right\rangle$ under gauge transformations in the standard way? Justify your answer mathematically. Also explain in a sentence or two why your answer makes physical sense.

b. Show that in the Heisenberg equation of motion for $\hat{\vec{v}}$ in component form is $m\dot{\hat{v}}_x^H = \frac{1}{2}q\left(\{\hat{v}_y^H, B_z(\hat{\vec{x}}^H)\} - \{\hat{v}_z^H, B_y(\hat{\vec{x}}^H)\}\right)$ where the dot indicates time derivatives (the equation for $\dot{\hat{v}}_y^H$ and $\dot{\hat{v}}_z^H$ are obtained via cyclic reorderings of x,y,z and need not be derived separately).

c. Suppose the magnetic field is independent of space and time. Take it to be oriented in the positive z direction and of magnitude B_0. Show that in this case, the Heisenberg equations of motion imply that $\hat{x}^H(t+\tau) = \hat{x}^H(t)$ $\hat{y}^H(t+\tau) = \hat{y}^H(t)$ $\hat{z}^H(t+\tau) = \hat{z}^H(t) + \tau\hat{v}_z^H(t)$ where $\tau = \dfrac{2\pi m}{qB_0}$. (It is interesting to note that τ is the period of a classical cyclotron orbit.)

d. The results in part c. clearly imply that $\langle y \rangle$ at time *t* is the same as $\langle y \rangle$ at time *t*+ τ.. Does it also mean that $\langle y^2 \rangle$ at time *t* is the same as $\langle y^2 \rangle$ at time *t*+ τ? Explain your answer.

5. This problem is a variation on a theme of coherent states. Coherent states are defined in terms of a particular harmonic oscillator basis. The harmonic oscillator considered here is a conventional one dimension harmonic oscillator for a particle of mass, *m*, and frequency, ω. We will define the raising and lowering operators ($\hat{a}^+$ and $\hat{a}$) for this problem in terms of the position and momentum in the standard way. Coherent states are normalized eigenstates of the lowering operator: $\hat{a}|z\rangle = z|z\rangle$ with $\langle z|z\rangle = 1$ where *z* is a complex number which labels the state. A coherent state can also be written as $|z\rangle = \exp\left(-\frac{1}{2}z^* z\right)\exp(z\hat{a}^+)|0\rangle$ where $|0\rangle$ is the ground state of the harmonic oscillator. In homework 5 you proved---or should have proved---the following identity for the overlap between two coherent states : $\langle z'|z\rangle = \exp\left(-\frac{1}{2}\left(|z|^2 + |z'|^2 - 2z'^* z\right)\right)$, which may prove useful.

This problem concerns two new classes of states related to coherent states:
$|z\rangle_E = \eta_E(z,z^*)\left(|z\rangle + |-z\rangle\right)$ $\quad |z\rangle_O = \eta_O(z,z^*)\left(|z\rangle - |-z\rangle\right)$ where $\eta_E(z,z^*)$ and $\eta_0(z,z^*)$ are normalization constants where the subscripts *E* and *O* denoted even and odd combinations.

a. Show that both $|z\rangle_E$ and $|z\rangle_O$ are eigenstates of $\hat{a}^2$. Find their eigenvalues.

b. Find the normalization constants $\eta_E(z,z^*)$ and $\eta_O(z,z^*)$.

c. Show that $|z\rangle_E$ and $|z\rangle_O$ can be expressed in the following form

$$|z\rangle_E = \sqrt{\frac{1}{\cosh(z^* z)}}\cosh(z\hat{a}^+)|0\rangle, \quad |z\rangle_O = \sqrt{\frac{1}{\sinh(z^* z)}}\sinh(z\hat{a}^+)|0\rangle.$$

d. Show that $\hat{a}|z\rangle_E = f(z^*,z)|z\rangle_O$ and $\hat{a}|z\rangle_O = g(z^*,z)|z\rangle_E$ where *f* and g are functions. Find the functional forms of *f* and g.

e. Evaluate the expectation value of x and x^2 in the state $|z\rangle_O$.