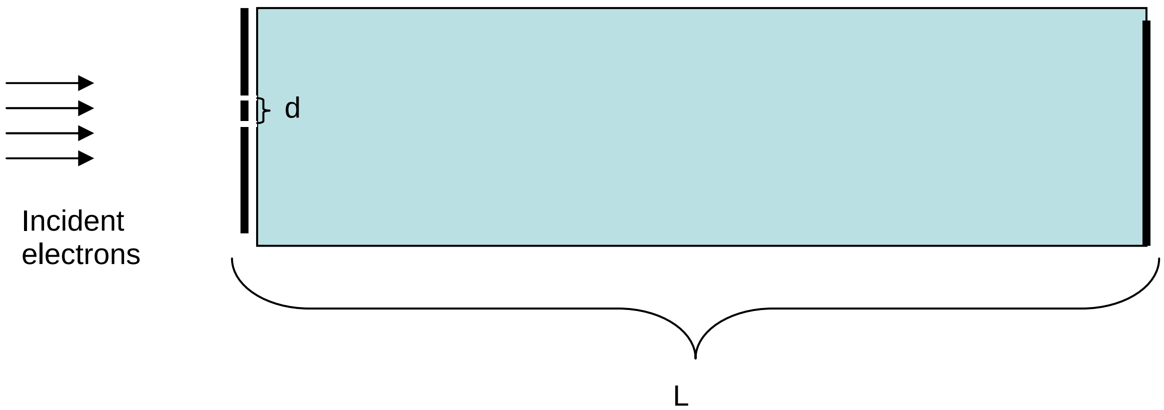
Homework 7: Due October 29

Note this assignment is longer than most. This is because it covers two weeks worth of material. Pleasebudget your time accordingly.

1. Consider a charged particle going through a double slit interference experiment where beyond the slit there is a constant magnetic field.



Where the magnetic field is in the blue region, has magnitude *B* and is directed out of the page, the incident electrons have a momentum of magnitude *p* and an interference pattern forms on the back plate. You may assume that *L* is very small compared with the radius of a classical cyclotron orbit of an electron of momentum *p* in a magnetic field of strength *B*; *i.e.* the classical path of an electron will not curve substantially as transverses a distance *L*. The maxima on this back plate shift due to the presence of the magnetic field. Show that the amount of this shift is equal to the amount a classical electron of this momentum is shifted upwards due to the Lorentz force as it travels from the slits to the back plate.

2. The current operator we derived in class $\hat{\vec{J}}(\vec{x}) = \frac{\hat{\vec{p}}\hat{\rho}(\vec{x}) + \hat{\rho}(\vec{x})\hat{\vec{p}}}{2m}$ was valid for Hamiltonians of the form $\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{x}})$. In this problem I want you to consider the case of a particle in an electro-magnetic field $\hat{H} = \frac{\left(\hat{\vec{p}} - q\vec{A}(\hat{\vec{x}})\right)^2}{2m} + q\phi(\vec{x})$. Find the appropriate form for $\hat{\vec{J}}(\vec{x})$ for this problem and show that current is conserved in the sense that $\frac{d\hat{\rho}^H(x)}{dt} = -\vec{\nabla}\cdot\hat{\vec{J}}^H(\vec{x})$. Hint: recall that for the classical version of the original problem the velocity is $\frac{\vec{p}}{m}$. What is the classical velocity in the new problem?

3. Consider a particle of charge *q* and mass *m* moving in a constant magnetic field in of strength *B* and oriented in the z direction. The purpose of this problem is find the

eigenstates of energy for this situation. These turn out to be the famous Landau levels. To proceed we need to pick a gauge. In this problem we will take gauge $\vec{A}(\vec{x}) = -B\,\hat{x}\,y$ where $\hat{x}$ is a unit vector in the x direction (and not an operator)

a. Show that this vector potential does in fact correspond the situation of interest.
b. Show that solutions to the the time-independent Schrodinger equation can be written as $\psi(x,y,z) = \exp\left(i\left(k_z z + k_x x\right)\right)\psi_n^{HO}\left(y - \frac{\hbar k_x}{qB}\right)$ where ψ_n^{HO} is the eigenfunction for the nth state of a one dimension harmonic oscillator for a system of mass *m* and $\omega = \left|\frac{qB}{m}\right|$ with associated energies given by

$$E_{k_z k_x n} = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega\left(n + \tfrac{1}{2}\right).$$

Note that the eigenstates are specified by two continuous parameters k_z, k_x and one discrete parameter *n*. Note also that the energies are completely independent of k_x and hence all level are infinite degenerate.

4. In a thermal ensemble the probability that a system will be in a particular energy eigenstate, *n*, is $P_n = \frac{\exp\left(-\beta E_n\right)}{Z(\beta)}$ where β and the partition function have the same meaning as in problem 1. The density matrix operator for given β is then given by $\hat{\rho}_\beta = \sum_n P_n \left|\psi_n\right\rangle\left\langle\psi_n\right| = \sum_n \frac{\exp\left(-\beta E_n\right)}{Z(\beta)}\left|\psi_n\right\rangle\left\langle\psi_n\right|$. Show that the position representation for the density matrix for system with one degree of freedom is given by $\rho_\beta(x',x) \equiv \left\langle x'\right|\hat{\rho}_\beta \,|\left|x\right\rangle$ is given by $\rho_\beta(x',x) = \frac{K^c(x',-i\beta\hbar;x,0)}{\int dx\; K^c(x,-i\beta\hbar;x,0)}$ where as in problem 1 $K^c(x,-i\beta\hbar;x,0)$ is understood in the sense of an analytic continuation from the real function.

5. Equation 3.3.21 in Sakurai gives an explicit matrix expression for a finite rotation specified the three Euler angles α, β , γ for a spin ½ system.
   a. Use the results in 1. to show that this rotation matrix may be represented as

$$\begin{aligned}\mathscr{D}(\alpha,\beta,\gamma) = &\left(\cos(\beta/2)\cos((\alpha+\gamma)/2)\right)\vec{\vec{1}} \\ &+ i\left(\left(\sin(\beta/2)\sin((\alpha-\gamma)/2)\right)\vec{\vec{\sigma}}_x - \sin(\beta/2)\cos((\alpha-\gamma)/2)\vec{\vec{\sigma}}_y - \cos(\beta/2)\sin((\alpha+\gamma)/2)\vec{\vec{\sigma}}_z\right)\end{aligned}$$

   b. Instead of the three Euler angles one can parameterize a general rotation as the rotation about a fixed axis $\hat{n}$ through a fixed angle, θ.

Use the result in a. to show that

$$\vartheta/2 = \cos^{-1}\left(\cos(\beta/2)\cos((\alpha+\gamma)/2)\right)$$

$$\hat{n} = -\frac{\left(\sin(\beta/2)\sin((\alpha-\gamma)/2)\right)\hat{x} - \sin(\beta/2)\cos((\alpha-\gamma)/2)\,\hat{y} - \cos(\beta/2)\sin((\alpha+\gamma)/2)\hat{z}_z}{\sin(\vartheta/2)}$$

where the hat on *n*,*x*, *y*, and *z* indicates that they are unit vectors (as opposed to operators.

6. In class we showed that the density matrix for a spin ½ system depended on three real parameters and was thus fully determined by the expectation values of the three components of spin: $\langle\langle s_x \rangle\rangle, \langle\langle s_y \rangle\rangle, \langle\langle s_z \rangle\rangle$. How many parameters does it take to fully specify the density matrix for a spin 1 system?

Sakurai----3.2, 3.4, 3.8