Homework 5 Due October 8

1. Prove the following identities for coherent states:
   a. $\langle z'|z\rangle = \exp\left(-\tfrac{1}{2}\left(|z|^2 + |z|^2 - 2z'^* z\right)\right)$
   b. $a^+|z\rangle = \left(\frac{\partial}{\partial z} + \frac{z^*}{2}\right)|z\rangle$
   c. Show that the identity operator may be written in terms of coherent states as $\hat{1} = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^\infty dr\, r|z\rangle\langle z|$ where $z = r\,e^{i\theta}$ (so that $|z\rangle \equiv |r\,e^{i\theta}\rangle$). Do this by evaluating $\left\langle n\left|\left(\frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^\infty dr\, r|r\,e^{i\theta}\rangle\langle r\,e^{i\theta}|\right)\right|m\right\rangle$ where $|m\rangle, |n\rangle$ are harmonic oscillator eigenstates. Hint: do the θ integral first.

2. Consider a three dimension Harmonic oscillator. The energy eigenstates are denoted $|n_x n_y n_z\rangle$.
   a. Evaluate the off-diagonal matrix element of the operator $\hat{\vec{J}}(\vec{x}) = \frac{\hat{\vec{p}}\hat{\rho}(\vec{x}) + \hat{\rho}(\vec{x})\hat{\vec{p}}}{2m}$ between the states $|000\rangle$ (*i.e.* the ground state) and the state $|100\rangle$. That is, find $\langle 0\ 0\ 0\,|\hat{\vec{J}}(\vec{x})|1\ 0\ 0\rangle$. Recall that $\hat{\rho}(\vec{x}) = |\vec{x}\rangle\langle\vec{x}|$.
   b. Find $\langle 000|\hat{\rho}(x)|100\rangle$
   c. Show explicitly that $\frac{\langle 000|[\hat{\rho}(x), \hat{H}]|100\rangle}{i\hbar} = -\vec{\nabla}\cdot\langle 000|\hat{\vec{J}}(\vec{x})|100\rangle$
   d. Suppose at *t=0* the system is in the state $|\psi(0)\rangle = \sqrt{\tfrac{3}{4}}|000\rangle + \tfrac{1}{2}|001\rangle$, find the expectation values $\langle\vec{J}(\vec{x})\rangle$ and $\langle\rho(\vec{x})\rangle$ as a function of time for t>0.
   e. Verify that $\partial_t\langle\rho(\vec{x})\rangle = -\vec{\nabla}\cdot\langle\vec{J}(\vec{x})\rangle$ for the expectation values in d.

3. Use the WKB quantization condition to find the semi-classical expressions for the energies of a one dimension harmonic oscillator.

4. The WKB expression for the wavefunction in the classical allowed region of a one dimensional system $\psi(x) = \frac{A}{\sqrt{k(x)}} \cos\left(\int_{x_{\min}}^{x} dx' \, k(x) - \frac{\pi}{4} \right)$ where $k(x) = \frac{\sqrt{2m\left(E - V(x)\right)}}{\hbar}$ where A is a normalization constant.

   a. Construct a semi-classical argument relating quantum mechanical probabilities to the time average probability to find the classical system at same position to show that the lowest order semi-classical estimate for A is $A \approx 2\sqrt{\frac{m}{\hbar \tau}}$ where τ is the period of the classical oscillation with the energy fixed.

   b. This choice of normalization constant does not yield a correctly normalized wavefunction if the wavefunction is considered in the classically allowed region. Explain why it does not. Explain why it is still a sensible normalization for the wave function despite this.

   c. Use the results in the previous sections to find the "normalized" WKB wavefunction for the n=20 state for the harmonic oscillator. Plot this against the exact n=20 harmonic oscillator level. You will need to use some computer package such as Mathematica to do this. Discuss the qualitative behavior both well into the classically allowed region far from the turning points and in the vicinity o the turning points.