Homework 12: Due December 10

1) Consider a spin-1 particle. In this problem you may assume that the dynamics of the spin degree of freedom is uncoupled from the position degree of freedom. This problem focuses on the spin and so it concerns a three-dimension vector space. We will use the standard basis $|m\rangle$ with $\hat{J}_z|m\rangle = m|m\rangle$, m=-1, 0, or 1 where $\hat{\vec{J}}$ is the spin. Assume that the Hamiltonian for this system is given by $\hat{H} = a\,\hat{J}_x^2$.

   a) Show that: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are energy eigenvectors and find the associated energy eigenvalues. You should find two degenerate eigenvalues.

   b) Show that $\hat{U}(t) = e^{-i\hat{H}t}$, the time evolution operator, can be written in matrix form as $\langle m'|\hat{U}(t)|m\rangle = \sum_{m''} d^{(1)+}_{m'm''}(\pi/2)\, d^{(1)}_{m''m}(\pi/2)\exp(-i\,a\,m''^2\,t)$, where $d^{(j)}_{m'm}(\theta) = \langle j\,m'|\exp(-i\hat{J}_y\theta)|\,j\,m\rangle$ is the Wigner "little d" matrix . *Hint: You may find it helpful to rewrite the Hamiltonian using the following identity:*
   $\hat{H} = \exp(i\hat{J}_y\pi/2)\left(\exp(-i\hat{J}_y\pi/2)\hat{H}\exp(i\hat{J}_y\pi/2)\right)\exp(-i\hat{J}_y\pi/2)$

   The time reversal properties of the system and its connection to the eigenstates of the system are of interest.

   c) Demonstrate that the Hamiltonian is invariant under time-reversal.

   d) Demonstrate that the non-degenerate energy eigenstate found in part a. is an eigenstate of the time reversal operator. *Hint: This can be done by general argument without reference to the specific form of the eigenvalue.*

   e) Two linear combinations of the degenerate energy eigenstates can be chosen which correspond to states that transform into one another under time reversal (up to possible phases). Find the appropriate linear combinations. Give a physical interpretation of these linear combinations in terms of eigenstates of $\hat{\vec{J}}\cdot\hat{n}$ about some axis and explain physically why this makes sense.

Sakurai **chapter 3: 24 ; Chapter 4: 7, 8b. , 11**