Homework 10: Due November 19

1. Clebsch-Gordan coefficients can be expressed in the form $\langle j_1 j_2 m_1 m_2 \mid jm\rangle$. Here I want you to compute from first principles all nonvanishing Clebsch-Gordan coefficients obtained from combining to angular momenta with j=1. That is you should calculate

$$\langle 1111|22\rangle \ \langle 1101|21\rangle \ \langle 1110|21\rangle \ \langle 11-11|20\rangle \ \langle 1100|20\rangle \ \langle 111-1|20\rangle \ \langle 11-10|2-1\rangle \ \langle 110-1|2-1\rangle \ \langle 11-1-1|2-2\rangle$$

$$\langle 1101|11\rangle \ \langle 1110|11\rangle \ \langle 11-11|10\rangle \ \langle 1100|10\rangle \ \langle 111-1|10\rangle \ \langle 11-10|1-1\rangle \ \langle 110-1|1-1\rangle$$

$$\langle 11-11|00\rangle \ \langle 1100|00\rangle \ \langle 111-1|00\rangle$$

Do this by noting that the state $|22\rangle = |1111\rangle$ and using the lowering operator and orthogonality. Alternatively you may use the recursion relation in the book. Compare your results with a C-G table.

2. Consider two Cartesian vectors $\vec{A}$ and $\vec{B}$. There are several ways to take products of the two to produce Cartesian tensors of various ranks: one can produce a scalar, namely the dot product $\vec{A}\cdot\vec{B}$ the cross product $\vec{A}\times\vec{B}$ and traceless symmetric tensor $\vec{C}$ whose components are given by $C_{ij} = \frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}\delta_{ij}\vec{A}\cdot\vec{B}$. Note that the components of these three structures are linearly independent: there are nine possible combination of $A_i B_j$ and there is one dot product, three components of the cross product and 5 independent components of C (once the fact that it is traceless and symmetric is included). This same information can be expressed in terms of three spherical tensor: an l=0, and l=1 and l=2. Derive the forms for these in terms of the Cartesian components A_i, B_j from first principles. In effect this is like deriving 3.10.16 but for a more general case Express Do not start from 3.10.16

3.

Consider the operator $\hat{\Theta} = \hat{J}_x^2 + \hat{J}_y^2 - 2\hat{J}_z^2$.

a. Show that $\hat{\Theta}$ can be written as the *q=0* component of a rank two spherical tensor, *i.e.* $\hat{\Theta} = \hat{T}_0^2$ .Find the general vector $\hat{T}_q^2$

b. Explicitly calculate the matrix element $\langle jm'|\hat{\Theta}|jm\rangle$ using general properties of angular momentum

c. Use the Wigner-Eckart theorem to show that $\dfrac{\langle jm_1|\hat{\Theta}|jm_1\rangle}{\langle jm_2|\hat{\Theta}|jm_2\rangle} = \dfrac{\langle j\,2\,m_1\,0|jm_1\rangle}{\langle j\,2\,m_2\,0|jm_2\rangle}$

d. Verify that formulae in c. satisfy the ratio's in b. for the case if *j*=2 and all *m*. It is sufficient to check the ratio of *m*=2 to *m*=1; *m*=1 to *m*=0; *m*=0 to *m*=-1; *m*=-1 to *m*=-2.

4.

The states of the three dimensional isotropic harmonic oscillator is often labeled by the Cartesian labels $\left|n_x n_y n_z\right\rangle$ these states are highly degenerate. It is possible to take linear combinations of these to obtain states with good *l* and *m*: $\left|n\,l\,m\right\rangle$. The purpose of this problem is to use tensorial properties to construct some of these $\left|n\,l\,m\right\rangle$ in terms of the $\left|n_x n_y n_z\right\rangle$ basis. We will work using creation operators for the three Cartesian directions

a. Show explicitly that $\hat{V}^1_\mu$ where $\hat{V}^1_0 = \hat{a}^+_z$ and $\hat{V}^1_{\pm 1} = \mp\frac{\left(\hat{a}^+_x \pm i\hat{a}^+_y\right)}{\sqrt{2}}$ is a rank one tensor by verifying it has the correct commutation rules with $\hat{\vec{L}}$.

b. Explicitly construct the tensor operators $\hat{S} = \sum_{\mu'}\left\langle 1 1 \mu' {-\mu'} \middle| 0 0\right\rangle \hat{V}^1_{\mu'}\hat{V}^1_{-\mu'}$,

$\hat{T}^2_2 = \sum_{\mu'}\left\langle 1 1 \mu'(\mu-\mu') \middle| 2 2\right\rangle \hat{V}^1_{\mu'}\hat{V}^1_{-\mu'}$, $\hat{T}^2_1 = \sum_{\mu'}\left\langle 1 1 \mu'(\mu-\mu') \middle| 2 1\right\rangle \hat{V}^1_{\mu'}\hat{V}^1_{-\mu'}$ and

$\hat{T}^2_0 = \sum_{\mu'}\left\langle 1 1 \mu'(\mu-\mu') \middle| 2 0\right\rangle \hat{V}^1_{\mu'}\hat{V}^1_{-\mu'}$ in terms of $\hat{a}^+_x, \hat{a}^+_y, \hat{a}^+_z$. By construction these are sphereical tensors.

c. Operate $\hat{V}^1_\mu$, $\hat{S}$ and $\hat{T}^2_\mu$ on the ground state and exploit the Wigner –Eckart theorem to find the states $\left|n\,l\,m\right\rangle = \left|111\right\rangle, \left|110\right\rangle, \left|11-1\right\rangle, \left|200\right\rangle, \left|222\right\rangle, \left|221\right\rangle, \left|220\right\rangle$ in terms of the states $\left|n_x n_y n_z\right\rangle$.