1. Consider a spin ½ particle in a magnetic field whose magnitude is constant but whose
direction rotates in the x-y plane with angular frequency $\Omega$. The Hamiltonian for such a
system can be written as $\hat{H} = -\omega (\hat{s}_x \cos(\Omega t) + \hat{s}_y \sin(\Omega t))$ where $\omega$ is a constant
which depends on the magnetic field and the gyromagnetic ratio. The purpose of this
problem is to find the time-evolution operator satisfying $\hat{H} \hat{U} = i \hbar \frac{d\hat{U}}{dt}$ with
$\hat{U}(0) = \hat{1}$.

a. As a first step show that $\hat{W} = \hat{W}^+(\omega \hat{s}_x)\hat{W}$ where $\hat{W}$ is a time dependent
unitary transformation given by $\hat{W} = \exp(i \hat{\sigma}_z \Omega t / 2)$. A couple of identities
you might want to prove first in deriving this are:
   i. $\{\hat{\sigma}_x, \hat{\sigma}_z\} = 0$ (anti-commutator0
   ii. $\{\hat{\sigma}_z, \hat{\sigma}_x\} = 2i \hat{\sigma}_y$
   iii. $\hat{\sigma}_z^2 = \hat{1}$

b. Show that $\hat{U}(t)$ can be written as $\hat{U}(t) = \hat{W}^+(t)\hat{V}(t)$ where $\hat{V}(t)$ satisfies an
effective time-evolution equation: $-(\omega \hat{s}_x + \Omega \hat{s}_z)\hat{V} = i \hbar \frac{d\hat{V}}{dt}$.

c. Show that $\hat{V} = \cos\left(\omega' t / 2\right)\hat{1} + i \sin\left(\omega' t / 2\right)\frac{\omega \hat{\sigma}_x + \Omega \hat{\sigma}_z}{\omega'}$ where
   $\omega' = \sqrt{\omega^2 + \Omega^2}$.

d. Use the expressions in b. and c. to find the matrix elements of $\hat{U}(t)$ in the
   $|+\rangle, |-\rangle$ basis.

2. In principle we know that the time evolution operator in problem 1 can be written as
$\hat{U}(t) = T\left(\exp\left(\frac{-i}{\hbar} \int_0^t dt' \hat{H}(t')\right)\right)$. Suppose we wish to approximate this by expanding the
exponential to second order: $\hat{U}(t) \approx \hat{1} + T\left(\frac{-i}{\hbar} \int_0^t dt' \hat{H}(t')\right) + \frac{1}{2} T\left(\frac{-i}{\hbar} \int_0^t dt' \hat{H}(t')\right)^2$.

a. Explain qualitatively why this should correspond to expanding the exact solution
up to second order in a Taylor series in $\omega$.

b. Compute the Taylor series for the exact solution up to second order in $\omega$. You
may find it VERY helpful to do this via Mathematica or other symbolic
manipulation program.

c. Calculate the time ordered products and compare with part b.

Sakurai---2.1, 2.3, 2.5., 2.6