622 Problem Set 3—Due Monday 9/24/07

1. Consider a spin ½ particle in a magnetic field whose magnitude is constant but whose direction rotates in the x-y plane with angular frequency Ω. The Hamiltonian for such a system can be written as $\hat{H} = -\omega\left(\hat{s}_x \cos(\Omega t) + \hat{s}_y \sin(\Omega t)\right)$ where ω is a constant which depends on the magnetic field and the gyromagnetic ratio. The purpose of this problem is to find the time-evolution operator satisfying $\hat{H}\hat{U} = i\hbar \frac{d\hat{U}}{dt}$ with $\hat{U}(0) = \hat{1}$.
   a. As a first step show that $\hat{H} = -\hat{W}^+(\omega \hat{s}_x)\hat{W}$ where $\hat{W}$ is a time dependent unitary transformation given by $\hat{W} = \exp\left(i\hat{\sigma}_z \Omega t/2\right)$. A couple of identities you might want to prove first in deriving this are:
      i. $\{\hat{\sigma}_x, \hat{\sigma}_z\} = 0$ (anti-commutator0
      ii. $[\{\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$
      iii. $\hat{\sigma}_z^2 = \hat{1}$
   b. Show that $\hat{U}(t)$ can be written as $\hat{U}(t) = \hat{W}^+(t)\hat{V}(t)$ where $\hat{V}(t)$ satisfies an effective time-evolution equation: $-\left(\omega\hat{s}_x + \Omega\hat{s}_z\right)\hat{V} = i\hbar\frac{d\hat{V}}{dt}$.
   c. Show that $\hat{V} = \cos\left(\omega' t/2\right)\hat{1} + i\sin\left(\omega' t/2\right)\frac{\omega\hat{\sigma}_x + \Omega\hat{\sigma}_z}{\omega'}$ where $\omega' = \sqrt{\omega^2 + \Omega^2}$.
   d. Use the expressions in b. and c. to find the matrix elements of $\hat{U}(t)$ in the $\left|+\right\rangle, \left|-\right\rangle$ basis.

2. In principle we know that the time evolution operator in problem 1 can be written as $\hat{U}(t) = T\left(\exp\left(\frac{-i}{\hbar}\int_0^t dt'\hat{H}(t')\right)\right)$. Suppose we wish to approximate this by expanding the exponential to second order: $\hat{U}(t) \approx \hat{1} + T\left(\frac{-i}{\hbar}\int_0^t dt' H(t')\right) + \frac{1}{2}T\left(\frac{-i}{\hbar}\int_0^t dt' H(t')\right)^2$.
   a. Explain qualitatively why this should correspond to expanding the exact solution up to second order in a Taylor series in ω.
   b. Compute the Taylor series for the exact solution up to second order in ω .You may find it VERY helpful to do this via Mathematica or other symbolic manipulation program.
   c. Calculate the time ordered products and compare with part b.

Sakurai---2.1, 2.3, 2.5., 2.6