

ELECTRODYNAMICS  
PROBLEM SET 4  
due February 23<sup>th</sup>, before class

**Problem 1.: Hidden momentum**

In the discussion of the energy flow in a coaxial cable in class we argue that the Poynting vector was non-vanishing inside the cable and pointed from the battery to the resistor. Since the momentum of the electromagnetic field is proportional to the Poynting vector, this result implies that the momentum of the electromagnetic fields is not zero. Since the total momentum of the charges in motion seem to be zero, this result would imply that the total momentum of the system is zero. For a static system, this is very odd ...

It turns out that the total mechanical momentum of the charges is not zero. In fact, it exactly cancels the electromagnetic one so the total momentum of the system is zero after all. The origin of this “hidden momentum” is a subtle relativistic effect so, instead of calculating it for the coaxial cable, we will use a simpler geometry.

Consider a square wire loop lying on the  $xy$  plane carrying a current  $I$  and immersed in an external constant electric field in the direction  $\hat{x}$ . The current generates a magnetic field that, combined with the external electric field, induces an electromagnetic momentum. In this problem we will show that there is a mechanical momentum in the currents going around the wire loop that cancels that.

a) Imagine that the current is carried by  $N_+$  charges moving at velocity  $v_+$  in one loop segment along  $\hat{y}$  and  $N_-$  charges with velocity  $v_-$  along the other segment along  $\hat{y}$ .  $v_+ \neq v_-$  because the charges are accelerated (decelerated) in the segments along  $\hat{x}$ . The current is

$$I = \frac{N_+ q v_+}{a} = \frac{N_- q v_-}{a}, \quad (1)$$

where  $q$  is the charge of each particle and  $a$  the size of the loop. What is the total mechanical (relativistic) momentum of the charges in the loop?

- b) What would the momentum be if we used non-relativistic mechanics instead?  
c) Write the mechanical momentum of the charges in terms of the magnetic dipole of the loop and the external electric field.

**Problem 2.: Multipole expansion**

Consider a linear distribution of charge along the  $z$ -axis where the charge density from  $z = 0$  to  $z = L/2$  is  $\lambda$  and from  $z = 0$  to  $z = -L/2$  is  $-\lambda$ .

- a) Calculate the potential by direct integration.  
b) Calculate the total charge, dipole moment and quadrupole moment of the charge distribution.  
c) Calculate the potential using the multipole expansion up to third (quadrupole) order.  
d) Compare the results in a) and c) by plotting the potential as a function of distance  $r$  along the  $x - y$  plane of the exact potential and the first, second and third order multipole approximations.  
e) Expand the analytic result in powers of  $L/r$  and verify that it agrees with the result in c).

**Problem 3: Pancakes and cigars**

Compute the quadrupole moment of

- a) a disk of radius  $a$  with surface charge density  $\sigma$  (a pancake).  
b) a line of length  $l$  and linear charge density  $\lambda$  (a cigar).

Hint: a lot of terms vanish by symmetry if a reasonable coordinate frame is used.

**Problem 4.: Multipoles in external fields**

Consider a static charge distribution with given total charge  $Q$ , electric dipole  $\vec{p}$  and quadrupole moment  $= Q^{ij}$  immersed in an electric field generated by *external fields* placed far away. Assume that the external electric field varies little in the region occupied by the charges.

- a) Compute the potential energy of the charges in the multipole expansion in terms of  $Q$ ,  $\vec{p}$ ,  $= Q^{ij}$ , the electric field and its derivatives at the charges location.  
b) From the result above, compute the force on the charge distribution  
c) Compute the torque on the charge distribution. Hint: the first two terms are obvious; I never calculated the quadrupole term.