CONSERVATION LAWS

CHARGE:

$$\underbrace{\partial_\nu \partial_\mu}_{\substack{\text{symmetric} \\ \text{under } \mu\leftrightarrow\nu}} \overbrace{F^{\mu\nu}}^{\substack{\text{anti-symmetric} \\ \text{under } \mu\leftrightarrow\nu}} = \partial_\nu \frac{4\pi}{c} J^\nu = 0 \quad \Rightarrow \quad \partial_\nu J^\nu = 0$$

$$\text{or} \quad \frac{1}{c}\frac{\partial}{\partial t} \underbrace{J^0}_{c\rho} + \vec{\nabla}\cdot\vec{J} = 0$$

or

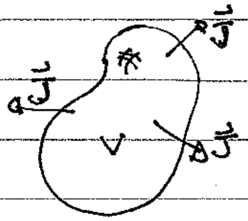
$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0$$

In integral form

$$\frac{\partial}{\partial t} \underbrace{\int_V d^3r\, \rho}_{Q} = \frac{\partial}{\partial t} Q = - \overbrace{\oint_{\partial V} da\, \hat{n}\cdot\vec{J}}^{\text{current flowing out of V}}$$

Q ← charge in volume V

∂V ← boundary of V



ENERGY-MOMENTUM:

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{g^{\mu\nu}}{4} F_{\alpha\beta}F^{\alpha\beta} \right]$$

$T^{\mu\nu}$ ← definition of the "energy-momentum-stress" tensor

$$\partial_\mu T^{\mu\nu} = \frac{1}{4\pi}\left[\underbrace{\partial_\mu F^{\mu}{}_{\alpha} F^{\alpha\nu}}_{\frac{4\pi}{c} F^{\alpha\nu} J_\alpha} + \underbrace{F^{\mu}{}_{\alpha}\, \partial_\mu F^{\alpha\nu}} + \frac{1}{4}\,\partial^\nu\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right]$$

$$F^{\mu}{}_{\alpha}\, \partial_\mu F^{\alpha\nu}$$

Bianchi identity

$$F_{\mu\alpha}\,\partial^\mu F^{\alpha\nu} \qquad\qquad -F_{\mu\alpha}\left(\partial^\nu F^{\mu\alpha} + \partial^\alpha F^{\nu\mu}\right)$$

$$= F_{\alpha\mu}\,\partial^\alpha F^{\mu\nu}$$

$$= F_{\mu\alpha}\,\partial^\alpha F^{\nu\mu}$$

$$\Downarrow$$

$$F^{\mu}{}_{\alpha}\, \partial_\mu F^{\alpha\nu} = \frac{1}{2} F_{\mu\alpha} \cancel{\partial^\alpha F^{\nu\mu}} + \frac{1}{2}\left(-F_{\mu\alpha}\right)\left(\partial^\nu F^{\mu\alpha} + \cancel{\partial^\alpha F^{\nu\mu}}\right)$$

$$= -\frac{1}{2} F_{\mu\alpha}\,\partial^\nu F^{\mu\alpha}$$

$$\Rightarrow \partial_\mu T^{\mu\nu} = \frac{1}{4\pi}\left[\frac{4\pi}{c} F^{\alpha\nu} J_\alpha - \frac{1}{2} F_{\mu\alpha}\,\partial^\nu F^{\mu\alpha} + \frac{1}{2}\underbrace{F_{\alpha\beta}\,\partial^\nu F^{\alpha\beta}}_{= F_{\mu\alpha}\,\partial^\nu F^{\mu\alpha}}\right]$$

$$= \frac{1}{c} F^{\alpha\nu} J_\alpha$$

$$\boxed{\partial_\mu T^{\mu\nu} = \frac{1}{c} F^{\alpha\nu} J_\alpha}$$

a regular conservation law would have a zero on the right hand side

Let us write $F^{\alpha\nu} J_\alpha$ in 3-D language and see what that means.

$$J_\nu F^{\mu\nu} \text{ for } \mu=0 \Rightarrow \underbrace{J_i F^{0i}}_{-F_{0i} = -(\vec{E})^i} + J_0 \cancelto{0}{F^{00}} = - \underbrace{\vec{J}.\vec{E}}_{\substack{\text{work/Time done} \\ \text{by the electric field} \\ \text{on the charges}}}$$

(magnetic field do no work)

That means that T^{00} should be the energy density of the E&M field and T^{0i} the energy flow of the E&M energy so

$$c \underbrace{\partial_0}_{\frac{1}{c}\frac{\partial}{\partial t}} \underbrace{T^{00}}_{\rho} + \underbrace{\partial_i c T^{0i}}_{\vec{\nabla}.\vec{S}} = - \underbrace{\vec{J}.\vec{E}}_{\substack{\text{powers Transmited} \\ \text{to the charges}}}$$

energy density

current flow of energy

Let us now write the components of $T^{\mu\nu}$ in 3D language To find out what the energy density ρ and current S are.

$$\underbrace{F_{\alpha\beta} F^{\alpha\beta}}_{- F_{\alpha\beta} F^{\beta\alpha} = - \mathrm{Tr}\, F^2} = - \mathrm{Tr} \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} = -2\vec{E}^2 + 2\vec{B}^2$$

$$T^{00} = \frac{1}{4\pi}\Big[-\underbrace{F^{0i}}_{-E^i}\underbrace{F^{i0}}_{E^i} + \frac{1}{4}2(\mathbb{B}^2-\mathbb{E}^2)\Big] = \frac{\mathbb{E}^2+\mathbb{B}^2}{8\pi} \longleftarrow \text{energy density}$$

$$T^{01} = \frac{1}{4\pi}\Big[-\underbrace{F^{0j}}_{-E^j}F^{j1}\Big] = \frac{1}{4\pi}\Big[E^2F^{21} + E^3F^{31} + E^1\overset{0}{\cancel{F^{11}}}\Big]$$

$$= \frac{1}{4\pi}\Big[E^2B^3 - E^3B^2\Big] = \left(\frac{\mathbb{E}\times\mathbb{B}}{4\pi}\right)^1$$

$$T^{02} = \dots$$

$$T^{03} = \dots$$

$$T^{0i} = \left(\frac{\mathbb{E}\times\mathbb{B}}{4\pi}\right)^i = \frac{1}{c}\underbrace{\left(\frac{c}{4\pi}\mathbb{E}\times\mathbb{B}\right)^i}_{\equiv S^i} \longleftarrow \text{Poynting vector}$$

$$\boxed{\frac{\partial}{\partial t}\underbrace{\left(\frac{\mathbb{E}^2+\mathbb{B}^2}{8\pi}\right)}_{\rho} + \nabla\cdot\underbrace{\left(\frac{c}{4\pi}\mathbb{E}\times\mathbb{B}\right)}_{\mathbb{S}} = -\mathbb{J}\cdot\mathbb{E}}$$

3D version of conservation of energy

We now look at the ν = spatial index components of $\partial_\mu T^{\mu\nu} = \frac{1}{c}J_\alpha F^{\alpha\nu}$, they will give us 3 more conservation relations. To find their meaning let us write them in 3D language.

$T^{\mu\nu}$ is symetric $T^{\mu\nu} = T^{\nu\mu}$

$$T^{io} = T^{oi} = \frac{1}{c}\left(\frac{c}{4\pi}\mathbb{E}\times\mathbb{B}\right)^i = \frac{1}{c}\mathbb{S}^i$$

$$T^{ij} = \frac{1}{4\pi}\Big[\underbrace{F^i{}_\alpha F^{\alpha j}}_{F^i{}_0 F^{0j} + F^i{}_k F^{kj}} + \underbrace{\frac{g^{ij}}{4}}_{-\frac{1}{4}\delta^{ij}}\underbrace{F_{\alpha\beta}F^{\alpha\beta}}_{-2(\mathbb{E}^2-\mathbb{B}^2)}\Big] = \frac{1}{4\pi}\left[-\mathbb{E}^i\mathbb{E}^j - \mathbb{B}^i\mathbb{B}^j + \delta^{ij}\mathbb{B}^2 + \frac{1}{2}\delta^{ij}(\mathbb{E}^2-\mathbb{B}^2)\right]$$

$$\underbrace{F^i{}_0 F^{0j} + F^i{}_k F^{kj}}_{\underbrace{F^{i0}F^{0j}}_{\mathbb{E}^i(-\mathbb{E}^j)} - \underbrace{F^{ik}F^{kj}}_{\underbrace{\begin{pmatrix} 0 & +B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}}_{\underbrace{\begin{pmatrix} -B^{3^2}-B^{2^2} & B^1B^2 & B^1B^3 \\ B^1B^2 & -B^{1^2}-B^{3^2} & B^2B^3 \\ B^1B^3 & B^2B^3 & -B^{1^2}-B^{2^2} \end{pmatrix}}_{\mathbb{B}^iB^j - \delta^{ij}\mathbb{B}^2}}}}$$

$$= \underbrace{\frac{1}{4\pi}\left[-\mathbb{E}^i\mathbb{E}^j - \mathbb{B}^i\mathbb{B}^j + \delta^{ij}\frac{\mathbb{E}^2+\mathbb{B}^2}{2}\right]}_{\equiv\, -T_M^{ij}}$$

$\longleftarrow$ Maxwell's stress tensor

$$\cancel{\frac{1}{c}J_0 F^{00} + \frac{1}{c}\underbrace{J_i}_{-\mathbb{J}^i}\underbrace{F^{i0}}_{\mathbb{E}^i}}$$

electric charge

$$\frac{1}{c}J_0 F^{01} + \frac{1}{c}J_j F^{j1} = \frac{1}{c}c\rho(-\mathbb{E}^1) + \frac{1}{c}(-\mathbb{J})^1\cdot\cancelto{0}{F^{11}} + \frac{1}{c}(-\mathbb{J})^2\underbrace{F^{21}}_{-F_{12}} + \frac{1}{c}(-\mathbb{J})^3\underbrace{F^{31}}_{-F_{13}}$$

$$= -\rho\mathbb{E}^1 - \frac{1}{c}J^2B^3 + \frac{1}{c}J^3B^2$$

$$= -\left(\rho\mathbb{E} + \frac{\mathbb{J}}{c}\times\mathbb{B}\right)^1$$

similarly $\frac{1}{c} J_\alpha F^{\alpha i} = -\underbrace{\left(\rho \mathbb{E} + \frac{\mathbb{J}}{c}\times\mathbb{B}\right)^i}_{\substack{\text{force density of the}\\ \text{E\&M field on the charges}}}$

so

$$\underbrace{\partial_0}_{\frac{1}{c}\frac{\partial}{\partial t}} \underbrace{T^{0i}}_{\frac{1}{c}S^i} + \underbrace{\partial_j T^{ji}}_{-\partial_j T_M^{ji}} = \underbrace{\frac{1}{c} J_0 F^{0i} + \frac{1}{c} J_j F^{ji}}_{-\left(\rho\mathbb{E} + \frac{\mathbb{J}}{c}\times\mathbb{B}\right)^i}$$

or

$$\boxed{\frac{\partial}{\partial t}\underset{\substack{\uparrow\\ \text{momentum}\\ \text{density of the}\\ \text{E\&M field}}}{\left(\frac{S}{c^2}\right)^i} = \underbrace{\nabla\cdot\overleftrightarrow{T}_M}_{\substack{\text{flow of}\\ \text{momentum}}} - \underbrace{\left(\rho\mathbb{E} + \frac{\mathbb{J}}{c}\times\mathbb{B}\right)}_{\substack{\text{E\&M force on}\\ \text{the charges}}}}$$

3D version
of conservation
of momentum

Now, let us get more physical.

The energy density in the electrostatic case (everything time-independent, no currents) can be written in a familiar way:

$$\underset{\substack{\uparrow\\ \text{energy density}}}{\rho_E} = \frac{1}{8\pi}\int_V d^3r\, \mathbb{E}^2 = \frac{1}{8\pi}\int d^3r\, (\nabla\phi)^2 = \frac{1}{8\pi}\int_V d^3r \left[\nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi\right]$$

$$= \frac{1}{8\pi}\oint_{\partial V} da\cdot\phi\nabla\phi + \frac{1}{8\pi}\int_V d^3r\, 4\pi\phi\underset{\nwarrow \text{ electric charge}}{\rho}$$

If $V\to\infty$: $\rho_E = \frac{1}{2}\int d^3r\, \rho\phi$

indexes the charges, not directions in space

For a set of discrete charges $q^{(i)}$ at positions $\mathbb{r}^{(i)}$.

$$PE = \frac{1}{2}\sum_i q^{(i)}\,\phi(\mathbb{r}^{(i)}) = \frac{1}{2}\sum_{ij} q^{(i)} \frac{q^{(j)}}{|\mathbb{r}^{(i)} - \mathbb{r}^{(j)}|}$$

$$= \sum_{i<j} \frac{q^{(i)}\, q^{(j)}}{|\mathbb{r}^{(i)} - \mathbb{r}^{(j)}|},$$

as you learned in high school. This is the energy required to bring these charges from infinity to their current position. Writting it in terms of the field suggests the energy is in the space, not in the charges. Only gravity cares about "where" the energy is though (in that case, "in the space" is the right answer). But wait a minute! $\mathbb{E}^2 > 0$ while $q^1q^2/|r^1-r^2|$ can have either sign. The resolution to this paradox is that $\frac{1}{8\pi}\mathbb{E}^2$ includes the energy to make the point charges while $q^1q^2/|r^1-r^2|$ doesn't. In the derivation above flaw was in not adding the self-energy of each point particle ($q^2/0$!).

The $\frac{\mathbb{B}^2}{8\pi}$ term also has a similar interpretation, what may be surprising as magnetic forces do no work. ~~It's still true, however, that currents attract/repel each other so it costs energy to assemble a distribution of currents.~~ But in order to set up a current distribution we have to increase the currents from zero to a given value, that is, you have to vary the current. Varying currents generate varying magnetic fields and that sets up an electric field that opposes the current (electromotive force). It requires energy to counter that. We can calculate it as

$$\underset{\substack{\text{work done by}\\\text{we to counter the e.m.f}}}{W} = \int dt \underset{\text{current}}{I} \underset{\substack{\text{integral}\\\text{around the}\\\text{circuit}}}{\oint d\mathbb{l}\cdot\mathbb{E}} = \int dt\, I \int d\mathbb{a}\cdot\nabla\times\mathbb{E} = \int dt\, I\, \frac{1}{c}\overset{\int d\mathbb{a}\cdot}{\frac{\partial}{\partial t}}\mathbb{B}$$

$$= \int dt\, I\, \frac{1}{c}\, \frac{\partial}{\partial t} \underbrace{\int da \cdot B}_{\Phi = LI} = \frac{1}{c} \int dI\, I = \frac{L}{2c} I^2$$

$$= \frac{1}{2c} I \int da \cdot B$$

flux is proportional to the current

proportionality constant (self-inductance)

We can think of any static distribution of currents as a superposition of loops (since $\nabla \cdot J = 0$, the integral lines of J are all closed). For this more general distribution

$$W = \frac{1}{2c} I \int da \cdot B = \frac{1}{2c} I \int da \cdot \nabla \times A = \frac{1}{2c} I \int dl \cdot A$$

$$\longrightarrow \frac{1}{2c} \int_V d^3r\; J \cdot A = \frac{1}{8\pi} \int_V d^3r\; A \cdot \nabla \times B = \frac{1}{8\pi} \int_V d^3r \left[B \cdot \nabla \times A - \nabla \cdot (A \times B) \right]$$

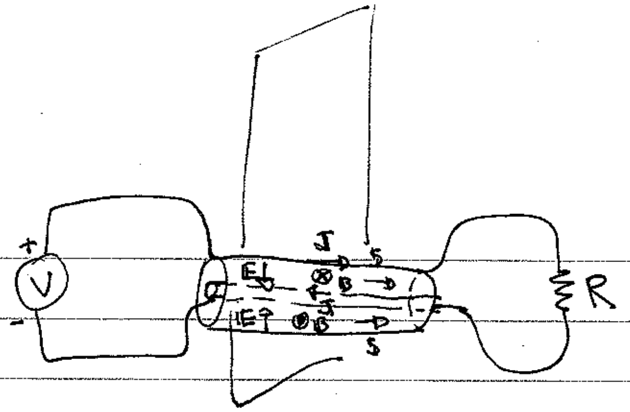
$$= \frac{1}{8\pi} \int_V d^3r\; B^2 - \frac{1}{8\pi} \oint_{\partial V} da \cdot A \times B$$

When $V \to \infty$ we find $W = \frac{1}{8\pi} \int d^3r\; B^2$.

These arguments should convince you that, in static configurations, $E = \frac{1}{8\pi} \int d^3r\, (E^2 + B^2)$. For the general case ~~[illegible] general [illegible]~~, see the general argument involving the Poynting vector, etc.

EXAMPLE:

energy flow in a circuit



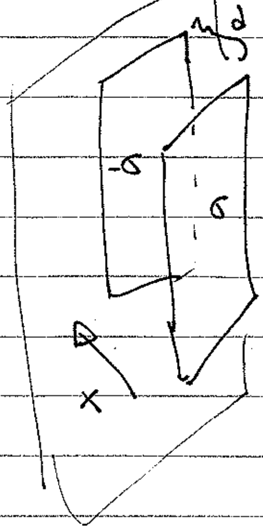
$$\nabla\times B = \frac{4\pi}{c}J \Rightarrow 2\pi r B = \frac{4\pi}{c} I \Rightarrow B = \frac{2I}{cr}$$

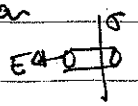
$$\nabla^2\phi = 0 \,,\ \phi(b) - \phi(a) = V \Rightarrow \phi = \frac{V}{\ln b/a}\ln r \Rightarrow E = \frac{V}{\ln b/a}\frac{1}{r}$$

$$S = \frac{1}{2\pi}\frac{VI}{\ln b/a}\frac{1}{r^2} \quad , \quad \int da \cdot S = \cancel{2\pi}\int_a^b dr\, r \frac{VI}{\cancel{2\pi}\ln b/a}\frac{1}{r^2} = \frac{VI}{\ln b/a}\ln r\Big|_a^b = VI$$

as standard circuit analysis would predict

EXAMPLE: force/area = pressure in a capacitor





Two plates

$$\nabla\cdot E = 4\pi\rho \Rightarrow aE = 4\pi a\sigma \Rightarrow E = 2\times 4\pi\sigma\,\hat{x}$$

$$T_M^{ij} = \frac{1}{4\pi}\left(E^iE^j - \frac{1}{2}\delta^{ij}E^2\right)$$

$$= \frac{1}{4\pi}(8\pi\sigma)^2\left(\hat{x}^i\hat{x}^j - \frac{1}{2}\delta^{ij}\right)$$

$$\frac{F^i}{\text{area}} = T_M^{ij}\hat{x}^j = 16\pi\sigma^2\underbrace{\left(\hat{x}^i - \frac{1}{2}\hat{x}^i\right)}_{\frac{1}{2}\hat{x}^i} = 8\pi\sigma^2, \text{ as expected.}$$

Physical interpretation of stress tensor :

In a fluid/solid (without e&m forces) the forces are short range and momentum is transfered by pushing and pulling adjacent volume elements. In that case

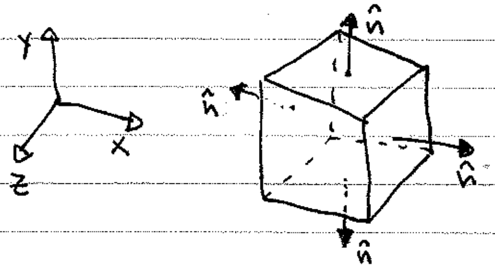
$$\vec{f} = \nabla\cdot\overleftrightarrow{T} \quad\Rightarrow\quad \vec{F} = \int_V d^3r\, \vec{f} = \int_V d^3r\, \nabla\cdot\overleftrightarrow{T} = \oint_{\partial V} da\, \hat{n}\cdot\overleftrightarrow{T}$$

force density

divergence of stress tensor

force on a volume V

boundary of V



$$\oint_{\partial V} da\, \hat{n}\cdot\overleftrightarrow{T} = a\Big[\hat{x}\cdot\overleftrightarrow{T}(\text{right}) - \hat{x}\cdot\overleftrightarrow{T}(\text{left}) + \hat{y}\cdot\overleftrightarrow{T}(\text{top}) - \hat{y}\cdot\overleftrightarrow{T}(\text{bottom}) + \hat{z}\cdot\overleftrightarrow{T}(\text{front}) - \hat{z}\cdot\overleftrightarrow{T}(\text{back})\Big] = \vec{F}$$

area of cube face

total force on the cube

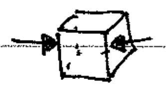
$$\Downarrow$$

$$\boxed{\hat{n}\cdot\overleftrightarrow{T} = \frac{\text{force}}{\text{area}} \text{ on the surface normal to } \hat{n}}$$

$$F^x = a\Big[\underbrace{\overbrace{T^{xx}(\text{right})}^{\text{pressure on the right}} - \overbrace{T^{xx}(\text{left})}^{\text{pressure on the left}}}_{\text{pressure gradient if } T^{xx}(\text{right}) \neq T^{xx}(\text{left})} + T^{yx}(\text{top}) - T^{yx}(\text{bottom}) + T^{zx}(\text{front}) - T^{zx}(\text{back})\Big]$$

x-component of The total force on V

shear



$$\boxed{\begin{array}{ll} T^{xx}, T^{yy}, T^{zz} = \text{pressure along } x,y,z \text{ directions} \\ T^{xy}, T^{yz}, T^{zx} = \text{shear} \end{array}}$$

<u>note 1</u>: In e&m the forces act at distance so $\oint_{\partial V} da\, \hat{n}\cdot\overleftrightarrow{T}$ gives the total e&m force on all particles inside V, even if the are not at the surface.

<u>note 2</u>: mass/energy is not a scalar, but one element of $T_{\mu\nu}$. In the relativistic theory of gravitation $T_{\mu\nu}$ plays the role of mass density as the source of gravitational fields

$$\nabla^2\phi = G\rho \longrightarrow R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$

That means that pressure and shear have their own gravity pull (or push).