# VECTORS, TENSORS, ...

## INTRODUCTION

You are familiar with Maxwell's equations

$$\nabla . \mathbb{E} = 4\pi\rho \qquad \nabla . \mathbb{B} = 0$$

$$\nabla \times \mathbb{B} - \frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} = \frac{4\pi}{c} \mathbb{j} \qquad \nabla \times \mathbb{E} + \frac{1}{c}\frac{\partial \mathbb{B}}{\partial t} = 0 .$$

They were discovered through painstaking experimentation:

Coulomb law

$$\nabla . \mathbb{E} = 4\pi\rho$$

absence of monopoles

$$\nabla . \mathbb{B} = 0$$

induction

$$\nabla \times \mathbb{E} + \frac{1}{c}\frac{\partial \mathbb{B}}{\partial t} = 0$$

$$\nabla \times \mathbb{B} - \frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} = \frac{4\pi}{c} \mathbb{j}$$

Maxwell figure this out just looking @ the equations

Ampère's law

Writting them down in components makes them look really arbitrary:

$$\frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial E_z}{\partial z} = 4\pi\rho$$

$$\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x + \frac{1}{c}\frac{\partial B_z}{\partial t} = 0$$

$$\vdots$$

Why should these particular combinations appear? It turns out that God had little choice in choosing those equations. After some general features of the Universe are chosen (mostly some symmetries) we are stuck with Maxwell's theory. In the 1st chapter we will explore that but, before, we need a little math. It's useful math anyway and, sooner or later, we will come across it.

VECTORS, TENSORS, ...

You are familiar with the concept of a vector. You probably picture them as something like:

$\vec{v}$

In order to do calculations we frequently use a basis, say, $\hat{x}, \hat{y}, \hat{z}$ (in this class, we will use always an orthonormal basis) to represent a vector as:

$$\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$$

components of $\vec{v}$ on the basis $\hat{x}, \hat{y}, \hat{z}$

What happens if I use a different orthonormal basis $\hat{x}', \hat{y}', \hat{z}'$. The <u>same</u> vector will have different components:

$$\vec{v} = v'_x \hat{x}' + v'_y \hat{y}' + v'_z \hat{z}'$$

components of $\vec{v}$ on the basis $\hat{x}', \hat{y}', \hat{z}'$

How are the components in one basis $(v_x, v_y, v_z)$ relate to the components in the other basis $(v'_x, v'_y, v'_z)$? In order to figure that out we first change notation a bit. We'll call the elements of the basis $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$ and $\hat{e}_3 = \hat{z}$ (similarly for the primed basis) and write:

$$\vec{V} = \sum_{i=1}^{3} v^i \hat{e}_i$$

Now, like any other vector, each one of the old basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ can be written in terms of the new basis.

$$\hat{e}_i = \sum_{j=1}^{3} (O^{-1})_i{}^j \, \hat{e}'_j \, ,$$

fancy name for the j-th component in the new basis of the i-th vector of the old basis

so

$$\vec{V} = \sum_{i=1}^{3} v^i \hat{e}_i = \sum_{j,i=1}^{3} v^i (O^{-1})_i{}^j \hat{e}'_j = \sum_{j=1}^{3} v'^j \hat{e}_j$$

and

$$v'^j = \sum_{i=1}^{3} v^i (O^{-1})_i{}^j \, .$$

We can think of the set of 9 number $(O^{-1})_i{}^j$ with $i,j = 1,2,3$ as a matrix

$$O^{-1} = \begin{pmatrix} O^{-1}_{11} & O^{-1}_{12} & O^{-1}_{13} \\ O^{-1}_{21} & O^{-1}_{22} & O^{-1}_{23} \\ O^{-1}_{31} & O^{-1}_{32} & O^{-1}_{33} \end{pmatrix} .$$

It turns out that the matrix $O^{-1}$ is orthogonal. In fact,

$$1 = |\hat{e}_1|^2 = |O^{-1}_{11}\hat{e}'_1 + O^{-1}_{12}\hat{e}'_2 + O^{-1}_{13}\hat{e}'_3| = (O^{-1}_{11})^2 + (O^{-1}_{12})^2 + (O^{-1}_{13})$$

and similarly for the other rows. Also,

$$0 = \hat{e}_1 \cdot \hat{e}_2 = (O^{-1}_{11}\hat{e}'_1 + O^{-1}_{12}\hat{e}'_2 + O^{-1}_{13}\hat{e}'_3) \cdot (O^{-1}_{21}\hat{e}'_1 + O^{-1}_{22}\hat{e}'_2 + O^{-1}_{31}\hat{e}'_3)$$

$$= O^{-1}_{11} O^{-1}_{21} + O^{-1}_{12} O^{-1}_{22} + O^{-1}_{13} O^{-1}_{31}$$

Those properties mean that $O^{-1}$ is an orthogonal matrix, that is,

$$O^{-1} = O^{T}$$

Transpose

so we can write,

on the O matrices the order of the indices matter ($O^i{}_j \neq O_j{}^i$) but being up or down doesn't ($O^i{}_j = O_i{}^j$)

$$\boxed{v'^j = \sum_{i=1}^{3} O^j{}_i \, v^i}$$

Sometimes is useful to think of these equations as matrix equations

first index indices the rows

$$\begin{pmatrix} v'^1 \\ v'^2 \\ v'^3 \end{pmatrix} = \begin{pmatrix} O^1{}_1 & O^1{}_2 & O^1{}_3 \\ O^2{}_1 & O^2{}_2 & O^2{}_3 \\ O^3{}_1 & O^3{}_2 & O^3{}_3 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

second index indices the columns

~~It turns out that we will very frequently have to consider~~

We say that "a vector transforms as" $v'^j = \sum_i O^j{}_i v^i$ but we should instead say "the components of a vector change under a change of basis" as $v'^j = \sum_i O^j{}_i v^i$. There'll be situations where we'll have 3 numbers defined for every orthonormal basis. The question then will be whether there is a vector whose components, in each basis, are those 3 numbers. The way to check is to verify $v'^j = \sum_i O^j{}_i v^i$. In fact, this relation is sometimes taken as a definition of a vector. We will consider examples of this but, first, let me generalize a little the concept of vector. We define the components of a 2-rank tensor as a set of 9 numbers $T^{ij}$ that change under a change of basis as

$$T'^{ij} = \sum_{k,l=1}^{3} O^i{}_k \, O^j{}_l \, T^{kl}$$

and similarly for higher rank tensors $T^{ijk}$, $T^{ijkl}$, ....

A single number $T$ (no indices) that doesn't change as the basis is changed is a rank zero tensor or, simply, a scalar. Let us look now at examples.

Scalars

The scalar product of two vectors $\vec{v}\cdot\vec{w} = \sum_{i=1}^{3} v^i w^i$. As we change the basis, the components $v^i$ and $w^i$ will change to $v'^i$ and $w'^i$. It's not obvious that the combination $\sum_i v^i w^i$ will remain the same. But it is true:

$$\sum_i v'^i w'^i = \sum_{k,i=1}^{3} \underbrace{O^i{}_j v^j}_{v'^i} \underbrace{O^i{}_k w^k}_{w'^i} = \sum_{i,k=1}^{3} \underbrace{(O^{-1})^j{}_i \, O^i{}_k}_{\substack{\text{multiplying row} \\ \text{by column = standard} \\ \text{matrix multiplication}}} v^j w^k$$

$$= \sum_{i,k=1}^{3} (O^{-1}O)^j{}_k \, v^j w^k$$

$$= \sum_{i,k=1}^{3} (\mathbb{1})^j{}_k \, v^j w^k$$

identity $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \sum_{i,k=1}^{3} \delta^j{}_k \, v^j w^k$$

Kronecker $\delta = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{if } j\neq k \end{cases}$

$$= \sum_{j=1}^{3} v^j w^j = \sum_{i=1}^{3} v^i w^i$$

Scalars are used in Physics to quantify temperature, energy, .... As an example of something that is not an scalar, consider $\sum_i a^i b^i c^i$, $a$, $b$, and $c$ being vectors. If we calculate this combination on a different basis we get a different number:

$$\sum_{i=1}^{3} a^i b^i c^i \neq \sum_{i=1}^{3} a'^i \, b'^i \, c'^i .$$

That explains why i) This triple product is not taught in school and ii) it can never appear in a physical law.

We can make scalars from "generalized scalar products, for instance

$$\sum_{i,j=1}^{3} T'^{ij} v'^i w'^j = \sum_{i,j,k,l,m,n=1}^{3} O^i{}_k \, O^j{}_l \, O^i{}_m O^j{}_n T^{kl} \; v^m \, w^n$$

$$= \sum_{i,j,k,l,m,n=1}^{3} \underbrace{O^i{}_k \left(O^{-1}\right)^m{}_i}_{\delta_k{}^m} \; \underbrace{O^j{}_l \left(O^{-1}\right)^n{}_j}_{\delta_l{}^n} \; v^m \, w^n \; T^{kl}$$

$$= \sum_{k,l=1}^{3} T^{kl} \, v^k \, w^l$$

An example of this is how we compute the energy of a spinning body by "contracting" the inertia tensor with the angular velocity (twice)

$$E = \frac{1}{2} \sum_{i,j} I^{ij} \omega^i \omega^j .$$

A similar argument shows that combinations like $\sum_{i,j,k,l=1}^{3} T^{ijk} \, v^k \, M^{jl} \, z^{il}$, where every index appears <u>twice</u> and are all summed over will be scalars too. For this reason we will use the convention (so-called "Einstein" convention) that the summation sign will be implicit every time the same index appears twice on an expression. So we write:

$$v'^i = O^i{}_j v^j , \qquad T^{ij} v^i w^j = V , \ldots$$

vectors

The vector product $\vec{v} \times \vec{w}$ of two vectors is also a vector. To show that its components indeed change under a basis change the way components of vectors do, let us first rewrite with index notation

$$\underbrace{(\vec{v} \times \vec{w})^i}_{i^{\text{th}} \text{ component of } \vec{v}\times\vec{w}} = \underbrace{\epsilon^{ijk}}_{\text{Levi-Civita symbol}} v^j w^k$$

$$\epsilon^{ijk} = \begin{cases} 1 & \text{for } i=1, j=2, k=3 \text{ or any cyclic permutation of this} \\ -1 & \text{for } i=2, j=1, k=3 \text{ or any cyclic permutation of this} \\ 0 & \text{if any indices are repeated} \end{cases}$$

$$= \begin{cases} \epsilon^{123} = \epsilon^{312} = \epsilon^{231} = 1 \\ \epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1 \\ \epsilon^{112} = \epsilon^{113} = \epsilon^{212} = \dots = 0 \end{cases}$$

Indeed, the $i=1$ component of $\vec{v}\times\vec{w}$ is $(\vec{v}\times\vec{w})^1 = \epsilon^{123} v^2 w^3 + \epsilon^{132} v^3 w^2 = v^2 w^3 - v^3 w^2$ and similarly for the other components. It turns out $\epsilon^{ijk}$ is not only a tensor but an "invariant tensor", one whose components are the same in any basis! The numerical value of the constants are defined above. It's a little tricky to see this. When we changed the basis the components of the Levi-Civita tensor should change as

$$\epsilon'^{ijk} = O^i{}_l O^j{}_m O^k{}_n \epsilon^{lmn}$$

The thing on the right, $O^i{}_l O^j{}_m O^k{}_n \epsilon^{lmn}$ seems formidable.

But it is simple. First, keep in mind it's something with 3 indices i, j and k (l, m, n are summed over). Second, it's antisymmetric by exchanging any of the 3 indices i, j, k.

$$O^j{}_l\, O^i{}_m\, O^k{}_n\, \epsilon^{lmn} = \underbrace{O^j{}_m\, O^i{}_l}\, O^k{}_n\, \epsilon^{mln}$$

I exchanged $i \leftrightarrow j$

I renamed $l \to m$ and vice-versa. It's a dummy index so it doesn't make a difference

$$= -\, O^i{}_l\, O^j{}_m\, O^k{}_n\, \epsilon^{lmn}$$

from changing $\epsilon^{mln}$ to $\epsilon^{lmn}$

Now, the antisymmetry property of $\epsilon'^{ijk}$ fixes all components up to an overall normalization. For instance, if $\epsilon'^{123} = 1$ then $\epsilon'^{213} = -1$, $\epsilon'^{312} = -\epsilon'^{132} = +\epsilon'^{123} = 1$, $\epsilon'^{112} = -\epsilon'^{112} = 0$ and so on. The only thing we need to fix is the normalization. A little thought shows that $\epsilon'^{123}$ is the determinant $\det O = 1$ (we are changing basis to another right-handed basis so $\det O = 1$). That all means that $\epsilon'^{ijk} = \epsilon^{ijk}$ and $\epsilon^{ijk}$ is, as claimed, an "invariant" tensor. Now it is easy to show that the vector product is a vector:

$$\epsilon'^{ijk}\, v'^j\, w'^k = \cancel{\epsilon\, O\, O^j{}_m\, O^k{}_n\, \ldots}$$

$$= O^i{}_r\, O^j{}_m\, O^k{}_n\, \epsilon^{rmn}\, O^j{}_l\, v^l\, w^p\, O^k{}_p$$

$$= \underbrace{O^j{}_m\, O^{-1\,l}{}_j}_{\delta_m{}^l}\; \underbrace{O^k{}_n\, O^{-1\,p}{}_k}_{\delta_n{}^p}\, O^i{}_r\, \epsilon^{imn}\, v^l w^p$$

$$= O^i{}_r\, \epsilon^{rmn}\, v^m w^n = O^i{}_r \left(\epsilon^{rmn} v^m w^n\right).$$

Similarly, one can show that "contracting" (= summing over) pairs of indices leaving one tensor "free" (= not summed over) generates a vector. For instance, the angular momentum of a solid body is

$$L^i = I^{ij} \omega^j$$

sum over a pair of identical indices

these components transform like the components of a vector, namely the angular momentum vector

tensor

vector

higher-rank tensors

"tensor product" of two vectors: $T^{ij} = v^i w^j$

The components $T^{ij}$ obviously transform like a rank-2 tensor:

$$T'^{ij} = v'^i w'^j = O^i{}_k O^j{}_l v^k w^l = O^i{}_k O^j{}_l T^{kl}$$

The electric quadrupole moment is an example of rank two tensor:

$$Q^{ij} = \int d^3x \, \rho(x) \left( 3 x^i x^j - x^2 \delta^{ij} \right)$$

by the way, can you show that the Kronecker $\delta$ is an (invariant) tensor?

## the $\vec{\nabla}$ operator as a vector

The nabla ( $\vec{\nabla}$ ) operator is not a vector but, under a change of basis, it transforms as the components of a vector. Indeed,

$$\frac{d}{dx'^i} = \frac{d}{dx^j}\underbrace{\frac{\partial x^j}{\partial x'^i}}_{O_i^{\ K} = O^i_{\ K}}$$

We will denote $\partial/\partial x^i$ by $\partial_i$ since it'll appear so often. Using the rule discussed above:

$$\boxed{\text{suming over pair of indices of tensors generate other Tensors}}$$

we know that

$\partial_i v^i = \nabla . v$ is a scalar ( the divergence)

$\epsilon^{ijk}\partial_j v^k = (\nabla \times v)^i$ is a vector ( the curl)

$\partial_i T^{ij}$ is a vector ( the divergence of a rank-2 tensor)

All relations you learned in vector calculus can be easily proved using components and index notations. But we'll leave that for the homework.