FIELD TRANSFORMATION RULES

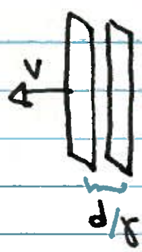
a) field of one charged surface

Gauss law ⇒ $2aE = 4\pi\,\overbrace{\underbrace{\sigma}_{\text{charge density}}\,a}^{\text{charge}}$

field due to the two plates : $\boxed{\begin{aligned} &E = 4\pi\sigma \\ &B = 0 \end{aligned}}$

no moving charges :

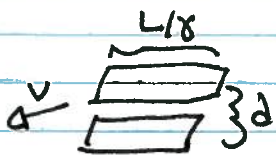
b)



charge density the same as in a). distance between plates Lorentz contracted ⇒ $\boxed{E' = 4\pi\sigma}$

moving charges produce a B field only at the boundary and we will neglect it. ⇒ $\boxed{B' = 0}$

c)

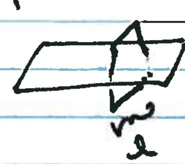


longitudinal size Lorentz contracted ⇒ $\sigma' = \gamma\sigma$ ⇒ $\boxed{\begin{aligned} E'' &= 4\pi\gamma\sigma \\ &= \gamma E \end{aligned}}$

current density $J = \sigma v$

field due to one moving charge sheet:

Ampère's law ⇒



$$2\ell B = \frac{4\pi}{c}\underbrace{\sigma' v}_{J}\,\ell = 4\pi\sigma\beta\gamma\ell$$

field due to the two moving plates: $\boxed{B'' = \frac{4\pi\sigma v\gamma}{c}}$

d) $F'_{\mu\nu} = L^{\rho}{}_{\nu} L^{\lambda}{}_{\mu} F_{\lambda\rho} = (L^T)^{\lambda}{}_{\mu} F_{\lambda\rho} L^{\rho}{}_{\nu}$

$$\Downarrow$$

$$\begin{pmatrix} 0 & E_x' & E_y' & E_z' \\ & 0 & -B_z' & -B_y' \\ \vdots & & 0 & -B_x' \\ & & & 0 \end{pmatrix} = \begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} 0 & E_x & 0 & E_z \\ -E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -E_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix}}_{\begin{pmatrix} -\beta\gamma E_z & E_x & 0 & \gamma E_z \\ -\gamma E_x & 0 & 0 & \beta\gamma E_x \\ 0 & 0 & 0 & 0 \\ -\gamma E_z & 0 & 0 & \beta\gamma E_z \end{pmatrix}}$$

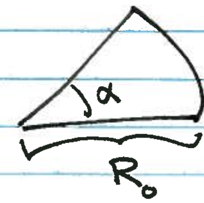
$$= \begin{pmatrix} 0 & \gamma E_x & 0 & \gamma^2 E_z - \beta^2\gamma^2 E_z \\ -\gamma E_x & 0 & 0 & \beta\gamma E_x \\ 0 & 0 & 0 & 0 \\ \beta^2\gamma^2 E_z - \gamma^2 E_z & -\beta\gamma E_x & 0 & -\beta\gamma^2 E_z + \beta\gamma^2 E_z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \gamma E_x & 0 & E_z \\ -\gamma E_x & 0 & 0 & \beta\gamma E_x \\ 0 & 0 & 0 & \\ -E_z & -\beta\gamma E_x & 0 & 0 \end{pmatrix}$$

In c) $E_x = 4\pi\sigma$, $E_z = 0$ so $E_x' = \gamma E_x$, $B_y = \beta\gamma\, 4\pi\sigma$

In b) $E_z = 4\pi\sigma$, $E_z' = \gamma\, 4\pi\sigma$ and $B = 0$
$E_x = 0$ as we calculated before.

BOUNDARY VALUE PROBLEM IN A PIZZA



we want to solve

$$\nabla^2\phi = 0$$
$$\phi(R_0,\theta) = V$$
$$\phi(r,0) = \phi(r,\alpha) = 0$$

in cylindrical coordinates:

$$\phi = R(r)\,\Theta(\theta) \quad \text{(no z dependence)}$$

$$\frac{1}{R}\frac{1}{r}(rR')' + \frac{1}{r^2}\frac{1}{\Theta}\Theta'' = 0 \qquad \Theta'' + m^2\Theta = 0 \qquad \text{(i)}$$

or

$$\underbrace{\frac{r}{R}(rR')'}_{m^2} + \underbrace{\frac{\Theta''}{\Theta}}_{-m^2} = 0 \qquad \Rightarrow \qquad (rR')' - m^2\frac{R}{r} = 0 \qquad \text{(ii)}$$

(i) $\Theta'' + m^2\Theta = 0 \Rightarrow \Theta(\theta) = \sin(m\theta),\ \cos(m\theta),\ \text{const},\ \theta$

(ii) $(rR')' - m^2\frac{R}{r} = 0 \Rightarrow R(r) = r^m,\ r^{-m},\ \ln r,\ \text{const.}$

boundary conditions:

$$\phi(r,0) = 0 \Rightarrow \Theta = \sin(m\theta)$$
$$\phi(r,\alpha) = 0 \Rightarrow \sin(m\alpha) = 0 \Rightarrow m = \frac{n\pi}{\alpha},\quad n = 1,2,\dots$$
$$\phi(0,\theta) = 0 \Rightarrow R = r^m = r^{n\pi/\alpha},\quad n = 1,2,\dots$$

$$\Downarrow$$

$$\phi(r,\theta) = \sum_{n=1}^{\infty} \underbrace{A_n}_{\text{constants to be determined}} r^{n\pi/\alpha} \overbrace{\sin\left(\frac{n\pi\theta}{\alpha}\right)\sqrt{\frac{2}{\alpha}}}^{\text{normalized basis}}$$

$$\phi(R_0,\theta) = V \Rightarrow \sum_{n=1}^{\infty} A_n R_0^{n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right)\sqrt{\frac{2}{\alpha}}$$

$$\Rightarrow A_n = \frac{1}{R_0^{n\pi/\alpha}}\int_0^{\alpha} d\theta\, V \underbrace{\sin\left(\frac{n\pi\theta}{\alpha}\right)\sqrt{\frac{2}{\alpha}}}_{\text{normalized basis}}$$

$$= \sqrt{\frac{2}{\alpha}}\frac{V}{R_0^{n\pi/\alpha}}\int_0^{n\pi} dx\, \frac{\alpha}{n\pi}\sin x$$

$$= \sqrt{\frac{2}{\alpha}}\frac{V}{R_0^{n\pi/\alpha}}\frac{\alpha}{n\pi}\begin{cases} +2, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$$

$$\phi(r,\theta) = \sum_{n\,odd} \frac{4V}{n\pi} \left(\frac{r}{R_0}\right)^{n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right)$$

This series can be solved analytically:

$$\text{define } f(\rho,\beta) = \sum_{n\,odd>0} \frac{\rho^n}{n} \sin(n\beta) = \sum_{\nu=0}^{\infty} \frac{\rho^{2\nu+1}}{2\nu+1} \sin((2\nu+1)\beta)$$

$$\frac{df}{d\rho} = \sum_{\nu=0}^{\infty} \rho^{2\nu} \sin((2\nu+1)\beta) = \frac{1}{2i} \sum_{\nu} \rho^{2\nu} \left(e^{i(2\nu+1)\beta} - e^{-i(2\nu+1)\beta}\right)$$

$$= \frac{1}{2i} \sum_{\nu=0}^{\infty} \left[e^{2\nu(i\beta+\ln\rho)} e^{i\beta} - e^{2\nu(-i\beta+\ln\rho)} e^{-i\beta} \right]$$

geometric series

$$= \frac{1}{2i} \left[e^{i\beta} \frac{1}{1-\rho^2 e^{2i\beta}} - e^{-i\beta} \frac{1}{1-\rho^2 e^{-2i\beta}} \right]$$

$$f = \int d\rho \frac{df}{d\rho} = \frac{1}{2i} \left[\cancel{e^{i\beta}}\, \cancel{e^{-i\beta}} \operatorname{arctgh}(e^{i\beta}\rho) - \cancel{e^{-i\beta}}\, \cancel{e^{i\beta}} \operatorname{arctgh}(e^{-i\beta}\rho) \right]$$

$$= \frac{1}{4i} \ln \Bigg(\underbrace{\frac{1-\rho e^{-i\beta}}{1+\rho e^{-i\beta}} \; \frac{1+\rho e^{i\beta}}{1-\rho e^{i\beta}}}_{\frac{1-\rho^2-2i\rho\sin\beta}{1-\rho^2+2i\rho\sin\beta}} \Bigg)$$

$$= \frac{1}{4i} \ln e^{2i \operatorname{arctg} \frac{2\rho\sin\beta}{1-\rho^2}}$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2\rho\sin\beta}{1-\rho^2}$$

($f(\rho=0,\beta)=0$ fixes the integration constant)

$$\boxed{\phi(r,\theta) = \frac{2V}{\pi} \operatorname{arctg}\left(\frac{2 r R_0 \sin(\pi\theta/\alpha)}{R_0^2 - r^2} \right)}$$

b) $\underbrace{\sigma}_{\text{induced charge density}} = \frac{E}{4\pi} = -\frac{|\nabla\phi|}{4\pi}$

$\underbrace{P}_{\text{pressure}} = \sigma E = \frac{E^2}{4\pi}$, $|E| = \frac{1}{r}\frac{\partial}{\partial\theta}\phi = \frac{4R_0}{\alpha}\frac{1}{R_0^2-r^2}$

$$\boxed{P = \frac{4}{\pi}\frac{R_0^2}{\alpha^2}\frac{1}{(R_0^2-r^2)^2}}$$

Maxwell-Chern-Simmons theory

a) $$\mathcal{L}_0 = +\frac{1}{4\pi}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\right] + j^\mu a_\mu$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu a_\nu} = \left[-\frac{1}{4}4F^{\mu\nu} + \frac{m}{2}\epsilon^{\alpha\mu\nu}a_\alpha\right]/4\pi\,, \quad \frac{\partial\mathcal{L}}{\partial a_\nu} = \left[\frac{m}{2}\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta + j^\nu 4\pi\right]\frac{1}{4\pi}$$

Euler-Lagrange eqs: $\partial_\mu\frac{\partial\mathcal{L}_0}{\partial\partial_\mu a_\nu} - \frac{\partial\mathcal{L}}{\partial a_\nu} = 0 \Rightarrow -\partial_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\alpha\mu\nu}\partial_\mu a_\alpha - \frac{m}{2}\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta + 4\pi j^\nu = 0$

$$\Rightarrow \quad \underbrace{\partial_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta - \frac{m}{2}\underbrace{\epsilon^{\alpha\mu\nu}}_{-\epsilon^{\nu\mu\alpha}}\partial_\mu a_\alpha}_{m\,\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta} = 4\pi j^\nu$$

$$\Rightarrow \quad \boxed{\partial_\mu F^{\mu\nu} + m\,\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta = 4\pi j^\nu}$$

b) In the Lorentz gauge $\partial_\mu a^\mu = 0$ and $j^\nu = 0$ the eq. of motion becomes

$$\partial_\mu(\partial^\mu a^\nu - \cancel{\partial^\nu a^\mu}^{\,0}) + m\,\epsilon^{\nu\alpha\beta}\partial_\alpha a_\beta = 0$$

The ansatz $a_\mu = A_\mu e^{ik\cdot x}$ implies that

$$-k_\mu k^\mu A^\nu + im\,\epsilon^{\nu\alpha\beta}k_\alpha A_\beta = 0$$

or

$$\left[-k^2 g^{\nu\beta} - im\,\epsilon^{\nu\beta\alpha}k_\alpha\right]A_\beta = 0$$

take $k_\alpha = (k_0, k_1, 0)$ w/o loss of generality

$$\underbrace{\begin{pmatrix} -k^2 & 0 & imk_1 \\ 0 & k^2 & -imk_0 \\ -imk_1 & imk_0 & k^2 \end{pmatrix}}_{\det = 0} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = 0$$

$$-(k^2)^3 \underbrace{- k^2 k_1^2 m^2 - k^2 k_0^2 m^2}_{-(k^2)^2 m^2} = 0 \Rightarrow k^2 = 0 \text{ or } k^2 = m^2$$