ELECTRODYNAMICS

PROBLEM SET 9

due April 26th, before class

I. MISSING STEP ON LIENARD-WIECHERT

In class we skipped the step from

$$F_{\mu\nu} = \frac{q}{(x-r(\tau)).v}\frac{d}{d\tau}\left[\frac{(x-r(\tau))_\mu v_\nu - (x-r(\tau))_\nu v_\mu}{(x-r(\tau)).v}\right]|_{ret.} \tag{1}$$

to

$$\mathbf{E} = \frac{q}{R^2}\frac{\hat{R}-\hat{\beta}}{\gamma^2(1-\beta.\hat{R})^3}|_{ret.} + \frac{q}{cR}\frac{1}{(1-\beta.\hat{R})^3}\hat{R}\times(\hat{R}-\hat{\beta})\times\dot{\beta}|_{ret.}, \tag{2}$$

$$\mathbf{B} = \hat{R}\times\mathbf{E}. \tag{3}$$

II. WAVEGUIDES

In this problem we will develop the basics of the theory of waveguides. Consider a hollow conductor with a constant cross section and infinite length (along the z-direction).

a) Using Maxwell's equations, show that, at the boundary

$$E_{\parallel} = 0 \;\text{ and }\; B_{\perp} = 0. \tag{4}$$

b) Consider the ansatz

$$\mathbf{E} = \mathcal{E}(x,y)e^{-i(\omega t - kz)}, \tag{5}$$

$$\mathbf{B} = \mathcal{B}(x,y)e^{-i(\omega t - kz)}, \tag{6}$$

describing a wave propagating down the pipe. What are the equations determining $\mathcal{E}_z, \mathcal{E}_\perp, \mathcal{B}_z, \mathcal{B}_\perp$ implied by Maxwell equations ?

c) Find $\mathcal{E}_\perp, \mathcal{B}_\perp$ in terms of the $\mathcal{E}_z, \mathcal{B}_z$ components. Using these equations we can determine the field inside the cavity if the z-components are given.

d) Show that the z-components satisfy the Helmholtz equation

$$\left[\nabla_\perp^2 - k^2 + \frac{\omega^2}{c^2}\right]\begin{cases}\mathcal{E}_z(x,y)\\ \mathcal{B}_z(x,y)\end{cases} = 0. \tag{7}$$

The problem now is reduced to a two-dimensional static problem. In general, the waves propagating on the guide will be a superposition of several modes. We can consider modes with $\mathcal{E}_z = 0$ (called TE modes, for transverse electric), $\mathcal{B}_z = 0$ (called TM modes) and even $\mathcal{E}_z = 0, \mathcal{B}_z = 0$ modes (TEM modes). The last one may seem like non-sense ($\mathcal{E}_z = \mathcal{B}_z = 0$) seems to imply that $\mathcal{E}_\perp = \mathcal{B}_\perp = 0$ by, in fact, they can exist if $\omega = kc$.

e) We now restrict ourselves to a pipe with a rectangular cross section of sides a and b. Solve the Helmholtz equation and find the TM modes allowed.

f) Find the frequency ω of the TM modes above as a function of k. Show that the phase velocity of these waves can larger than c but that the group velocity is smaller than c

g) Sketch the electric and magnetic fields for the lowest TM mode inside the cavity. *By sketching I mean using all you got, pencil, computer graphics and movies to convey the geometry of the fields. Last year I got some excellent movies that I'll post on the webpage.*

With small changes on the boundary condition the ideas developed here can describe dielectric cables (optical fibers), coaxial cables, microstrips, old two-wire cables (like those used to connect an antenna to a tv set), the intercontinental communication of whales, stethoscopes, etc ...