Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 8.1 Drawing with Crayons

Compute the TE and TM modes in a rectangular cavity. Then, to the best of your abilities, make a sketch of the field lines in both cases. You are encouraged to use computer graphics, ray tracing, ... The best one gets posted to the course webpage.

Solution

See Jackson 8.4.

Problem 8.2 If the Qualifier were this week, would you pass it?

An electromagnetic field

$$\mathbf{E} = \Re\left(E_y(x,z)e^{-i\omega t}\hat{\mathbf{y}}\right)$$

propagates through a medium with dielectric constant $\epsilon(x)$ and permeability $\mu = 1$.

a. Derive a differential equation (from Maxwell's equations) for the complex amplitude $E_y(x,z)$.

b. Obtain expressions for the magnetic field and verify that the Maxwell equations are satisfied.

c. Show that solutions that are nearly plane waves ($E_y(x,z) = \tilde{E}_y(x,z)e^{ik_0 z}$ with $k_0\tilde{E}_y \gg \partial\tilde{E}_y/\partial z$) satisfy an equation of the form,

$$2ik_0\frac{\partial}{\partial z}\tilde{E}_y(x,z) + \frac{\partial^2}{\partial x^2}\tilde{E}_y(x,z) - V(x)\tilde{E}(x,z) = 0,$$

which is analogous to the Schrödinger equation. What plays the role of the potential $V(x)$?

d. If the medium is homogeneous (ϵ =constant) a gaussian shaped wave spread in x as it propagates in z.

$$\tilde{E}_y(x,z) = \frac{E_0}{\sqrt{\pi\sigma(z)}}e^{-\frac{x^2}{\sigma(z)}}.$$

Obtain an expression for the z-dependent complex width σ in terms of the initial value $\sigma(0)$ and the wave number k_0 for the special value of k_0 such that $V = 0$.

e. How must the dielectric constant depend on x and what must be the wave number k_0 be in order that the wave does not spread as it propagates?

Solution

Part a.

If we assume time dependence as given in the equation for $\mathbf{E}$ at the start of the problem, Maxwell's equations become

$$\nabla\cdot\mathbf{D} = 0, \qquad \nabla\times\mathbf{H} + i\tfrac{\omega}{c}\mathbf{D} = 0,$$
$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} - i\tfrac{\omega}{c}\mathbf{B} = 0.$$

Since $\mu = 1$ we can relate $\mathbf{H} = \mathbf{B}$, and if we look at the divergence of $\mathbf{D}$ we obtain

$$\nabla\cdot(\epsilon(x)\mathbf{E}) = \frac{\partial\epsilon(x)}{\partial x}E_x + \epsilon(x)\nabla\cdot\mathbf{E},$$

but $E_x = 0$, so we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0.$$

Now if we use this along with the curl of $\mathbf{E}$ equation above,

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) - i\tfrac{\omega}{c}\boldsymbol{\nabla} \times \mathbf{B} &= 0 \\ \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \left(\tfrac{\omega}{c}\right)^2 \epsilon(x)\mathbf{E} &= 0 \\ -\nabla^2 \mathbf{E} - \left(\tfrac{\omega}{c}\right)^2 \epsilon(x)\mathbf{E} &= 0. \end{aligned}$$

Now, if we impose that that $\mathbf{E}$ is only in the y direction and only depends on x and z, then we get (letting $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$):

$$\left(-\nabla_\perp^2 - \left(\tfrac{\omega}{c}\right)^2 \epsilon(x)\right) E_y(x, z) = 0. \tag{1}$$ {eq:1}

Part b.

Consider the Maxwell equation

$$i\tfrac{\omega}{c}\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{E},$$

to solve this question. Obviously $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and putting this into the curl of $\mathbf{B}$ equation just reproduces part a.

Part c.

Now, we take $E_y = \tilde{E}_y e^{ik_0 z}$ with the condition $k_0 \tilde{E}_y \gg \frac{\partial \tilde{E}_y}{\partial z}$ and if we take the second derivative we have

$$\frac{\partial^2 E_y}{\partial z^2} = \left(\frac{\partial^2 \tilde{E}_y}{\partial z^2} + 2ik_0 \frac{\partial \tilde{E}_y}{\partial z} - k_0^2 \tilde{E}_y\right) e^{ik_0 z} \sim \left(2ik_0 \frac{\partial \tilde{E}_y}{\partial z} - k_0^2 \tilde{E}_y\right) e^{ik_0 z}.$$

This expression can be used in Eq. (1) immediately and we obtain

$$\left[-\frac{\partial^2}{\partial x^2} - 2ik_0 \frac{\partial}{\partial z} + \left(k_0^2 - \epsilon(x)\tfrac{\omega^2}{c^2}\right)\right] \tilde{E}_y = 0,$$

so we have $V(x) = k_0^2 - \epsilon(x)\omega^2/c^2$. This is the expression we were to find.

Part d.

Now if we take a constant ϵ, we have $V = 0$, we plug in the form of the Gaussian. We need for that:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left(\frac{E_0}{\sqrt{\pi\sigma(z)}} e^{-\frac{x^2}{\sigma(z)}}\right) &= \frac{E_0}{\sqrt{\pi\sigma(z)}}\left[\left(\frac{2x}{\sigma(z)}\right)^2 - \frac{2}{\sigma(z)}\right] e^{-\frac{x^2}{\sigma(z)}}, \\ \frac{\partial}{\partial z}\left(\frac{E_0}{\sqrt{\pi\sigma(z)}} e^{-\frac{x^2}{\sigma(z)}}\right) &= \frac{E_0}{\sqrt{\pi\sigma(z)}}\left[-\frac{\sigma'(z)}{2\sigma(z)} + x^2 \frac{\sigma'(z)}{\sigma(z)^2}\right] e^{-\frac{x^2}{\sigma(z)}}. \end{aligned}$$

Using these we can get an equation for $\sigma(z)$

$$\frac{2}{\sigma(z)} - \left(\frac{2x}{\sigma(z)}\right)^2 + ik_0\frac{\sigma'(z)}{\sigma(z)} - 2ik_0x^2\frac{\sigma'(z)}{\sigma(z)^2} = 0$$
$$(\sigma(z) - 2x^2)(2 + ik_0\sigma'(z)) = 0.$$

Thus, since σ is independent of z, we must have

$$\sigma(z) = \frac{2iz}{k_0} + \sigma(0).$$

Part e.

For the wave to not spread as it propagates we must have $\sigma(z) = \sigma$ independent of z, so the z-derivative will vanish and we can still use the second derivative computed above which will yield the formula

$$-\frac{4x^2}{\sigma^2} + \frac{2}{\sigma} + V(x) = 0,$$

so we have

$$V(x) = \frac{4x^2}{\sigma^2} - \frac{2}{\sigma}.$$

But we know that $V(x) = k_0^2 - \epsilon(x)\omega^2/c^2$, so that implies that we need (choosing the constants to make it nicer)

$$\epsilon(x) = A - B^2x^2,$$

and thus, we have

$$k_0^2 - \frac{A\omega^2}{c^2} + \frac{B^2x^2\omega^2}{c^2} = \frac{4x^2}{\sigma^2} - \frac{2}{\sigma}.$$

Since the above is true for all x, we must have

$$\frac{2}{\sigma} = \frac{B\omega}{c},$$

so we can find what k_0 is:

$$k_0^2 = \frac{\omega}{c}\left(A - B\frac{\omega}{c}\right).$$

We can clearly see that we need $\frac{A}{B} > \frac{\omega}{c}$ so that we get wave propagation.