Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 7.1 Reflection with Dispersion

A plan wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a complex $n(\omega)$. Show that the ratio of reflected power to incident power is

$$R = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2 .$$

Solution

Even though the coefficient is complex, the derivation of section 7.3 of Jackson works so for normal incidence we can apply Eq. (7.42) of Jackson for

$$\frac{E_0''}{E_0} = \frac{1 - n(\omega)}{1 + n(\omega)},$$

where The power is then obtained from the real part of the Poynting vector (note we let $\mu = 1$ for convenience)

$$\mathbf{S} = \frac{c}{8\pi} \mathbf{E} \times \mathbf{H}^{\star} = \frac{c}{8\pi} \left| \mathbf{E} \right|^2 \hat{\mathbf{k}}.$$

Now, we have for the reflection coefficient

$$R = \frac{\Re(\hat{\mathbf{n}} \cdot \mathbf{S}_0'')}{\Re(\hat{\mathbf{n}} \cdot \mathbf{S}_0)} = \left| \frac{E_0''}{E_0} \right|^2 = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2 .$$

Problem 6.2/7.2 Faraday Rotation

In class we considered a simple model for dielectrics where the motion of the electron was approximated by charged, damped harmonic oscillators. In this problem we extend that model by adding a small external constant magnetic field B_0.

a. Write down the (non-relativisitic) equation of motion of a harmonic oscillator under the influence of the electric field of a plane wave and the external, constant B_0 field. Argue that the influence of the magnetic field of the plane wave is small as long as the electron is non-relativistic. Contrary to what was done in class, neglect any damping.

b. Assuming that the B_0 field is in the direction of motion of a circularly polarized plane wave, solve the equations of motion for the electron. *Hint*: Fourier transform, just like in class.

c. Calculate the polarization density and the dielectric constant as a function of the frequency ω. Notice that they depend on the polarization of the wave. This phenomena is called *birefringence*.

d. Show that the polarization direction of a linearly polarized wave will change as the wave propagates along the direction of B_0.

Hint: You should find that, for dilute plasmas ($n \approx 1, \omega_0 \approx 0$), the rate of rotation is proportional to the density of free electrons and it is faster for low frequency waves. These facts, combined, allow astronomers to measure density and magnetic fields in far away regions of space.

Solution

Part a.

Using Newton's law, we have the equation of motion given as

$$m\ddot{\mathbf{x}} = -m\Omega^2\mathbf{x} + q\left(\mathbf{E}e^{-i\omega_0 t} + \tfrac{1}{c}\dot{\mathbf{x}} \times \mathbf{B}_0\right).$$

We neglect the effect of the magnetic part of the wave since we assume B_0 is much larger than it, and as long it stays non-relativistic, the magnetic force will remain on the same order as the electric force.

Part b.

First, we note that there is no force in the $\hat{\mathbf{k}}$ direction, so we can assume $\mathbf{x}$ only moves in the polarization plane, and thus can be written as a combination of $\mathbf{x} = x_+\boldsymbol{\epsilon}_+ + x_-\boldsymbol{\epsilon}_-$ (with circular polarizations defined in part b of 6.1). If we have $\mathbf{E}$ circularly polarized and $\mathbf{B}$ in the direction of propogation, then

$$\mathbf{E} = E\boldsymbol{\epsilon}_\pm,$$
$$\mathbf{B} = B_0\hat{\mathbf{k}}$$

At this point, we Fourier transform the equation of motion to get

$$-m(\omega^2 - \Omega^2)\mathbf{x}(\omega) = q\left[E\boldsymbol{\epsilon}_\pm \delta(\omega - \omega_0) + i\tfrac{\omega}{c} B_0\, \mathbf{x}(\omega) \times \hat{\mathbf{k}}\right].$$

Now, we note that $\boldsymbol{\epsilon}_\pm \times \hat{\mathbf{k}} = \pm i \boldsymbol{\epsilon}_\pm$, so $\mathbf{x}$, so x_+ decouples from x_-. Thus, which ever is not in line with the polarization of the wave is just zero! Thus, we have

$$-m(\omega^2 - \Omega^2)\mathbf{x}(\omega) = q\left[\mathbf{E}_\pm \delta(\omega - \omega_0) \mp \tfrac{\omega}{c} B_0\, \mathbf{x}(\omega)\right].$$

Solving this, we have (setting $q = -e$)

$$\mathbf{x}(\omega) = \frac{e\,\mathbf{E}_\pm}{m\left[\omega(\omega \pm eB_0/mc) - \Omega^2\right]}\delta(\omega - \omega_0).$$

In position space (letting $\omega_0 = \omega$), we obtain

$$\mathbf{x} = \frac{eE}{m\left[\omega(\omega \pm \omega_B) - \Omega^2\right]}\boldsymbol{\epsilon}_\pm e^{-i\omega t},$$

where we have let $\omega_B = eB_0/mc$.

Part c.

The polarization for a single particle is then

$$\mathbf{p} = -\frac{e^2\mathbf{E}}{m\left[\omega(\omega \pm \omega_B) - \Omega^2\right]}e^{-i\omega t}.$$

Now, if there are n molecules per unitvolume and Z electrons per molecule, then we have approximately the average polarization is

$$\mathbf{P} = -\frac{nZe^2}{m\left[\omega(\omega \pm \omega_B) - \Omega^2\right]}\mathbf{E}.$$

Then we have $\epsilon(\omega)\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$, so we can read off directly

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B) - \Omega^2},$$

where $\omega_p^2 = 4\pi nZe^2/m$ is the plasma frequency. The $+$ sign is for right-handed waves and $-$ is for left-handed waves.

Part d.

A linear polarized wave will begin with a (real) polarization, so we have

$$\boldsymbol{\epsilon} = A\boldsymbol{\epsilon}_+ + A^\star \boldsymbol{\epsilon}_-.$$

For purpose of illustration, we will choose $A = A^\star = 1$. These two parts evolve differently since we have two different dispersion relations for the two different circular polarizations:

$$k_\pm^2 = \epsilon_\pm(\omega)(\omega/c)^2.$$

Thus, we have

$$\mathbf{E} = E_0\left(\boldsymbol{\epsilon}_+ e^{i\mathbf{k}_+\cdot\mathbf{x}} + \boldsymbol{\epsilon}_- e^{i\mathbf{k}_-\cdot\mathbf{x}}\right)e^{-i\omega t}.$$

To see that the polarization is changing directions (because clearly it does not stay the same), we take $\omega_B \ll \omega$ (and $\Omega \ll \omega_B$ for convenience) so that we can write

$$\begin{aligned}
k_\pm &= \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2(1 \pm \omega_B/\omega)}} \\
&\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2} \pm \frac{\omega_B\omega_P^2}{\omega^3}} \\
&\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\sqrt{1 \pm \frac{\omega_B}{\omega}\frac{\omega_P^2}{\omega^2 - \omega_P^2}} \\
&\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\left(1 \pm \frac{\omega_B}{2\omega}\frac{\omega_P^2}{\omega^2 - \omega_P^2}\right) \\
&\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \pm \frac{\omega_B}{2c}\frac{\omega_P^2/\omega^2}{\sqrt{1 - \omega_P^2/\omega^2}}.
\end{aligned}$$

Thus, we have that $k_\pm = k_0 \pm k'$ (in this limit), and we can write our electric field as

$$\mathbf{E} = E_0\left(\boldsymbol{\epsilon}_+ e^{i\mathbf{k}'\cdot\mathbf{x}} + \boldsymbol{\epsilon}_- e^{-i\mathbf{k}'\cdot\mathbf{x}}\right)e^{i\mathbf{k}_0\cdot\mathbf{x} - i\omega t}.$$

Thus, we can see that the polarization remains linear (i.e. notice the polarization is still *real*), but it changes direction based on how far it goes in the medium (for example, if $\mathbf{x} = 0$, we get $\boldsymbol{\epsilon}_1$ and if $\mathbf{x}\cdot\mathbf{k}' = \pi/2$ we get $-\boldsymbol{\epsilon}_2$).

Clearly the rate of rotation is given by this k', and for dilute plasmas this is given by

$$k' = \frac{\omega_B}{2c}\frac{\omega_p^2}{\omega^2}.$$

Clearly for lower frequencies, this rate is faster, and it is faster if there are more electrons per unit volume (which raises ω_p).