Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:
$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 6.1 Spin of Electromagnetic Wave

The angular momentum of the electromagnetic field is defined as

$$\mathbf{L} = \frac{1}{4\pi c}\int \mathrm{d}^3 r \; \mathbf{r} \times (\mathbf{E} \times \mathbf{B}),$$

and is a conserved quantity in the absence of matter (you proved this in a previous homework).

a. Consider a field distribution localized in space and show that $\mathbf{L}$ can be split as

$$\mathbf{L} = \frac{1}{4\pi c}\int \mathrm{d}^3 r \left[\mathbf{E} \times \mathbf{A} + E_i(\mathbf{r} \times \boldsymbol{\nabla})A_i\right].$$

The first term is called the spin (since it doesn't depend on the origin of the coordinate system) and the second one is the orbital angular momentum.

b. Compute the spin for a linearly, right-handed and left-handed circularly polarized plane wave. (A plane wave is not a localized field distribution but a similar result would follow from considering a localized wave packet.)

Solution

Part a.

To tackle this problem, we will have a total of 3 cross products to handle. Thus, we look at the integrand in component notation:

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \epsilon_{ijk}\epsilon_{klm} r_j E_l B_m \\
&= \epsilon_{ijk}\epsilon_{klm}\epsilon_{mnp} r_j E_l \partial_n A_p \\
&= \epsilon_{ijk}(\delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln}) r_j E_l \partial_n A_p \\
&= \epsilon_{ijk} r_j E_l(\partial_k A_l - \partial_l A_k) \\
&= E_l(\epsilon_{ijk} r_j \partial_k) A_l - \epsilon_{ijk} r_j E_l \partial_l A_k.
\end{aligned}$$

From the above calculation we see that the first term is one of the terms that we want, and on the second term, we recall that we are integrating *and* the field is localized, so if we integrate by parts, the surface term will vanish. Doing the integration by parts we get under the integral:

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &\to E_l(\mathbf{r} \times \boldsymbol{\nabla})_i A_l + \epsilon_{ijk}\partial_l(r_j E_l) A_k \\
&= E_l(\mathbf{r} \times \boldsymbol{\nabla})_i A_l + \epsilon_{ijk} E_j A_k + \epsilon_{ijk} r_j (\partial_l E_l) A_k.
\end{aligned}$$

In the last term of the immediate above, we assume there is no charge, so $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. Thus we have arrived at the result

$$\mathbf{L} = \frac{1}{4\pi c}\int \mathrm{d}^3 r \left[\mathbf{E} \times \mathbf{A} + E_i(\mathbf{r} \times \boldsymbol{\nabla})A_i\right].$$

Part b.

For a linearly polarized wave we have (in phasor space, so the physical part of this is the real part)

$$\mathbf{E} = E_0 \boldsymbol{\epsilon} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t},$$

where $\boldsymbol{\epsilon}$ is real, and since $\mathbf{B} = \hat{\mathbf{k}} \times \mathbf{E} = i\mathbf{k} \times \mathbf{A}$, we can choose $\mathbf{A} = ik\mathbf{E}$. Using this we have the spin density

$$\begin{aligned}
\frac{1}{4\pi c}\Re\{\mathbf{E}\} \times \Re\{\mathbf{A}\} &= \frac{1}{16\pi c}\left(\mathbf{E} + \mathbf{E}^\star\right) \times \left(\mathbf{A} + \mathbf{A}^\star\right) \\
&= \frac{ik}{16\pi c}\left(\mathbf{E} + \mathbf{E}^\star\right) \times \left(\mathbf{E} - \mathbf{E}^\star\right) \\
&= 0.
\end{aligned}$$

For ciruclarly polarized waves we have (in phasor space, so the physical part of this is the real part)

$$\mathbf{E}_\pm = E_0 \boldsymbol{\epsilon}_\mp e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t},$$

where the + is for right-handed waves and − is for left-handed waves. In the above we have used the definition

$$\boldsymbol{\epsilon}_\pm = \tfrac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2),$$

where $\boldsymbol{\epsilon}_{1,2}$ are the two orthogonal polarizations. Some properties of interest:

$$\begin{aligned}
\boldsymbol{\epsilon}_\pm^\star &= \boldsymbol{\epsilon}_\mp, \\
\boldsymbol{\epsilon}_\pm^\star \times \boldsymbol{\epsilon}_\pm &= \pm i\hat{\mathbf{k}}, \\
\boldsymbol{\epsilon}_\pm \times \boldsymbol{\epsilon}_\pm &= 0.
\end{aligned}$$

Then we can find

$$\mathbf{B}_\pm = \hat{\mathbf{k}} \times \mathbf{E}_\pm$$

Note that we can find $\mathbf{A}$ by $\mathbf{B} = i\mathbf{k} \times \mathbf{A}$, so we can easily see that

$$\mathbf{A}_\pm = ik\mathbf{E}_\pm.$$

Using this, then we can find the spin density (we use the above properties of $\boldsymbol{\epsilon}_\pm$)

$$\begin{aligned}
\frac{1}{4\pi c}\Re\{\mathbf{E}_\pm\} \times \Re\{\mathbf{A}_\pm\} &= \frac{1}{16\pi c}\left(\mathbf{E}_\pm + \mathbf{E}_\pm^\star\right) \times \left(\mathbf{A}_\pm + \mathbf{A}_\pm^\star\right) \\
&= \frac{ik}{16\pi c}\left(\mathbf{E}_\pm + \mathbf{E}_\pm^\star\right) \times \left(\mathbf{E}_\pm - \mathbf{E}_\pm^\star\right) \\
&= \frac{ik}{16\pi c}\left[-\mathbf{E}_\pm \times \mathbf{E}_\pm^\star + \mathbf{E}_\pm^\star \times \mathbf{E}_\pm\right] \\
&= \frac{ik}{16\pi c}\mathbf{E}_\pm^\star \times \mathbf{E}_\pm \\
&= E_0^2 \frac{ik}{8\pi c}\boldsymbol{\epsilon}_\mp^\star \times \boldsymbol{\epsilon}_\mp \\
&= \pm\frac{E_0^2 \mathbf{k}}{8\pi c}.
\end{aligned}$$

Thus, we have found the spin-density for both of these waves:

$$\frac{1}{4\pi c}\Re\left\{\mathbf{E}\right\}\times\Re\left\{\mathbf{A}\right\}=\frac{E_0^2\mathbf{k}}{8\pi c}\begin{cases}+1, & \text{right-handed circularly polarized waves,}\\ 0, & \text{linearly polarized waves,}\\ -1, & \text{left-handed circularly polarized waves.}\end{cases}$$

Problem 6.2/7.2 Faraday Rotation

In class we considered a simple model for dielectrics where the motion of the electron was approximated by charged, damped harmonic oscillators. In this problem we extend that model by adding a small external constant magnetic field B_0.

a. Write down the (non-relativisitic) equation of motion of a harmonic oscillator under the influence of the electric field of a plane wave and the external, constant B_0 field. Argue that the influence of the magnetic field of the plane wave is small as long as the electron is non-relativistic. Contrary to what was done in class, neglect any damping.

b. Assuming that the B_0 field is in the direction of motion of a circularly polarized plane wave, solve the equations of motion for the electron. *Hint*: Fourier transform, just like in class.

c. Calculate the polarization density and the dielectric constant as a function of the frequency ω. Notice that they depend on the polarization of the wave. This phenomena is called *birefringence*.

d. Show that the polarization direction of a linearly polarized wave will change as the wave propagates along the direction of B_0.

Hint: You should find that, for dilute plasmas ($n \approx 1, \omega_0 \approx 0$), the rate of rotation is proportional to the density of free electrons and it is faster for low frequency waves. These facts, combined, allow astronomers to measure density and magnetic fields in far away regions of space.

Solution

Part a.

Using Newton's law, we have the equation of motion given as

$$m\ddot{\mathbf{x}} = q\left(\mathbf{E}e^{-i\omega_0 t} + \tfrac{1}{c}\dot{\mathbf{x}} \times \mathbf{B}_0\right).$$

We neglect the effect of the magnetic part of the wave since we assume B_0 is much larger than it, and as long it stays non-relativistic, the magnetic force will remain on the same order as the electric force.

Part b.

First, we note that there is no force in the $\hat{\mathbf{k}}$ direction, so we can assume $\mathbf{x}$ only moves in the polarization plane, and thus can be written as a combination of $\mathbf{x} = x_+\boldsymbol{\epsilon}_+ + x_-\boldsymbol{\epsilon}_-$ (with circular polarizations defined in part b of 6.1). If we have $\mathbf{E}$ circularly polarized and $\mathbf{B}$ in the direction of propogation, then

$$\mathbf{E} = E\boldsymbol{\epsilon}_\pm,$$
$$\mathbf{B} = B_0\hat{\mathbf{k}}$$

At this point, we Fourier transform the equation of motion to get

$$-m\omega^2\mathbf{x}(\omega) = q\left[E\boldsymbol{\epsilon}_\pm\delta(\omega-\omega_0) + i\tfrac{\omega}{c}B_0\,\mathbf{x}(\omega)\times\hat{\mathbf{k}}\right].$$

Now, we note that $\boldsymbol{\epsilon}_\pm \times \hat{\mathbf{k}} = \pm i\boldsymbol{\epsilon}_\pm$, so $\mathbf{x}$, so x_+ decouples from x_-. Thus, which ever is not in line with the polarization of the wave is just zero! Thus, we have

$$-m\omega^2\mathbf{x}(\omega) = q\left[\mathbf{E}_\pm\delta(\omega-\omega_0) \mp \tfrac{\omega}{c}B_0\,\mathbf{x}(\omega)\right].$$

Solving this, we have (setting $q=-e$)

$$\mathbf{x}(\omega) = \frac{e\,\mathbf{E}_\pm}{m\omega(\omega \pm eB_0/mc)}\delta(\omega-\omega_0).$$

In position space (letting $\omega_0=\omega$), we obtain

$$\mathbf{x} = \frac{eE}{m\omega(\omega\pm\omega_B)}\boldsymbol{\epsilon}_\pm e^{-i\omega t},$$

where we have let $\omega_B = eB_0/mc$.

Part c.

The polarization for a single particle is then

$$\mathbf{p} = -\frac{e^2\mathbf{E}}{m\omega(\omega\pm\omega_B)}e^{-i\omega t}.$$

Now, if there are n molecules per unitvolume and Z electrons per molecule, then we have approximately the average polarization is

$$\mathbf{P} = -\frac{nZe^2}{m\omega(\omega\pm\omega_B)}\mathbf{E}.$$

Then we have $\epsilon(\omega)\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$, so we can read off directly

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega\pm\omega_B)},$$

where $\omega_p^2 = 4\pi nZe^2/m$ is the plasma frequency. The $+$ sign is for right-handed waves and $-$ is for left-handed waves.

Part d.

A linear polarized wave will begin with a (real) polarization, so we have

$$\boldsymbol{\epsilon} = A\boldsymbol{\epsilon}_+ + A^\star\boldsymbol{\epsilon}_-.$$

For purpose of illustration, we will choose $A = A^\star = 1$. These two parts evolve differently since we have two different dispersion relations for the two different circular polarizations:

$$k_\pm^2 = \epsilon_\pm(\omega)(\omega/c)^2.$$

Thus, we have

$$\mathbf{E} = E_0 \left(\boldsymbol{\epsilon}_+ e^{i\mathbf{k}_+ \cdot \mathbf{x}} + \boldsymbol{\epsilon}_- e^{i\mathbf{k}_- \cdot \mathbf{x}} \right) e^{-i\omega t}.$$

To see that the polarization is changing directions (because clearly it does not stay the same), we take $\omega_B \ll \omega$ so that we can write

$$\begin{aligned} k_\pm &= \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2(1 \pm \omega_B/\omega)}} \\ &\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2} \pm \frac{\omega_B \omega_P^2}{\omega^3}} \\ &\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\sqrt{1 \pm \frac{\omega_B}{\omega}\frac{\omega_P^2}{\omega^2 - \omega_P^2}} \\ &\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\left(1 \pm \frac{\omega_B}{2\omega}\frac{\omega_P^2}{\omega^2 - \omega_P^2}\right) \\ &\sim \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \pm \frac{\omega_B}{2c}\frac{\omega_P^2/\omega^2}{\sqrt{1 - \omega_P^2/\omega^2}}. \end{aligned}$$

Thus, we have that $k_\pm = k_0 \pm k'$ (in this limit), and we can write our electric field as

$$\mathbf{E} = E_0 \left(\boldsymbol{\epsilon}_+ e^{i\mathbf{k}' \cdot \mathbf{x}} + \boldsymbol{\epsilon}_- e^{-i\mathbf{k}' \cdot \mathbf{x}} \right) e^{i\mathbf{k}_0 \cdot \mathbf{x} - i\omega t}.$$

Thus, we can see that the polarization remains linear (i.e. notice the polarization is still *real*), but it changes direction based on how far it goes in the medium (for example, if $\mathbf{x} = 0$, we get $\boldsymbol{\epsilon}_1$ and if $\mathbf{x} \cdot \mathbf{k}' = \pi/2$ we get $-\boldsymbol{\epsilon}_2$).

Clearly the rate of rotation is given by this k', and for dilute plasmas this is given by

$$k' = \frac{\omega_B}{2c}\frac{\omega_p^2}{\omega^2}.$$

Clearly for lower frequencies, this rate is faster, and it is faster if there are more electrons per unit volume (which raises ω_p).