Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 1. Angular Momentum

One point that was not emphasized in class is that conservation laws are a consequence of symmetries of Nature. For instance, translation symmetry leads to the conservation of momentum and energy, gauge invariance leads to conservation of charge, ... Rotation invariance, being a symmetry of the electromagnetic action, should lead to the existence of a conserved quantity: angular momentum.

a. Show that the angular momentum tensor $M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$ satisfy $\partial_\lambda M^{\mu\nu\lambda} = 1/c(x^\mu J_\alpha F^{\alpha\nu} - x^\nu J_\alpha F^{\alpha\mu})$.

b. Write down the equation derived above in 3-dimensional notation and describe the physical interpretation of each term.

Solution

Part a.

The angular momentum tensor is given as $M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$, so we have

$$\begin{aligned}\partial_\lambda M^{\mu\nu\lambda} &= \partial_\lambda(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}) \\ &= T^{\mu\nu} + x^\mu \partial_\lambda T^{\nu\lambda} - T^{\mu\nu} - x^\nu \partial_\lambda T^{\mu\lambda} \\ &= x^\mu \partial_\lambda T^{\nu\lambda} - x^\nu \partial_\lambda T^{\mu\lambda}.\end{aligned}$$

In the above, we have used the symmetric version of the energy momentum tensor (see Jackon pp. 605–610 for its construction):

$$T^{\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\alpha}F_{\alpha\lambda}F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right).$$

Also, note that Maxwell's equations (two of the four) can be written as

$$\partial_\lambda F^{\lambda\mu} = \frac{4\pi}{c}J^\mu.$$

Using this, it is a straightforward exercise to establish

$$\partial_\lambda T^{\lambda\mu} = -\tfrac{1}{c}F^{\mu\alpha}J_\alpha. \tag{1}$$

We can return to our first calculation now:

$$\begin{aligned}\partial_\lambda M^{\mu\nu\lambda} &= \tfrac{1}{c}\left(-x^\mu F^{\nu\alpha}J_\alpha + x^\nu F^{\mu\alpha}J_\alpha\right) \\ &= \tfrac{1}{c}\left(x^\mu J_\alpha F^{\alpha\nu} - x^\nu J_\alpha F^{\alpha\mu}\right).\end{aligned}$$

The last line above follows from anti-symmetry of $F^{\mu\nu}$, and we have arrived at the result we needed.

Part b.

First, let us deal with some hairy math that is not very enlightening. We need to use the following facts (see the previous homework solutions, Eq. (1) and Eq. (2) for the fact about the electromagnetic field F):

$$\begin{aligned} T^{00} &= u, \\ T^{i0} &= S_i/c, \\ T^{ij} &= -T^{(M)}_{ij}, \\ F^{i0} &= E_i, \\ F^{ij} &= -\epsilon_{ijk}B_k. \end{aligned}$$

We have used u for the energy-density $(E^2+B^2)/8\pi$, $\mathbf{S}$ for the Poynting vector, and $T^{(M)}$ for the Maxwell stress tensor.

It is hard to interpret the $\partial_\lambda M^{0i\lambda}$ component since it is essentially some “rotation” in a time-spatial plane (much like an x and a y corresponds to rotation in the xy-plane). Although from the analysis, you can find the usual energy-momentum conservation equations for the e/m field.

That aside, we concentrate on $\partial_\lambda M^{ij\lambda}$ and take special note of the following facts:

$$M^{ij\lambda} = x^i T^{j\lambda} - x^j T^{i\lambda} = (\delta^{il}\delta^{jm} - \delta^{jl}\delta^{im})x^l T^{m\lambda} = \epsilon^{kij}\epsilon^{klm}x^l T^{m\lambda}.$$

Now, if we look at $\frac{1}{2}\epsilon^{nij}M^{ij\lambda}$, we see that we just get: $\frac{1}{2}\epsilon^{nij}M^{ij\lambda} = \epsilon^{nlm}x^l T^{m\lambda}$, a cross product of sorts! In the same exact fashion it can be shown that

$$\tfrac{1}{2c}\epsilon^{nij}\left(x^i J_\alpha F^{\alpha j} - x^j J_\alpha F^{\alpha i}\right) = \tfrac{1}{c}\epsilon^{nlm}x^l J_\alpha F^{\alpha m},$$

so we can put this together to get that the equation derived in part a implies

$$\begin{aligned} \partial_\lambda(\epsilon^{nlm}x^l T^{m\lambda}) &= \tfrac{1}{c}\epsilon^{nlm}x^l J_\alpha F^{\alpha m} \\ \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{S})_n - \frac{\partial}{\partial x^k}\left(\epsilon^{nlm}x^l T^{(M)}_{mk}\right) &= -\epsilon^{nlm}x^l\left(\rho E^m + \epsilon_{mkj}J^k B_j\right) \\ \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{S}) - \hat{\mathbf{x}}_n\frac{\partial}{\partial x^k}\left(\epsilon^{nlm}x^l T^{(M)}_{mk}\right) &= -\mathbf{r}\times(\rho\mathbf{E} + \tfrac{1}{c}\mathbf{J}\times\mathbf{B}). \end{aligned}$$

If we re-arrange the above, we get

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{S}) + \mathbf{r}\times(\rho\mathbf{E} + \tfrac{1}{c}\mathbf{J}\times\mathbf{B}) = \hat{\mathbf{x}}_n\frac{\partial}{\partial x^k}\left(\epsilon^{nlm}x^l T^{(M)}_{mk}\right). \tag{2}$$

Now, if we integrate this over some volume, we see that the term on the right-hand side is a divergence, so by Gauss's theorem we can write it as an integral over the surface of our volume (with normal n_k). Thus, we can write

$$\int \mathrm{d}^3x\,\hat{\mathbf{x}}_n\frac{\partial}{\partial x^k}\left(\epsilon^{nlm}x^l T^{(M)}_{mk}\right) = \oint_S \hat{\mathbf{x}}_n\epsilon^{nlm}x^l T^{(M)}_{mk}n_k\,\mathrm{d}A$$

As for the other terms, note that $(\rho\mathbf{E} + \frac{1}{c}\mathbf{J}\times\mathbf{B}) = \frac{\partial\mathbf{p}}{\partial t}$ where $\mathbf{p}$ is the momentum *density*. Then, the last term is $\frac{\partial}{\partial t}\mathbf{r}\times\mathbf{p}$, the time derivative of the angular momentum *density*. If we integrate over the volume, we simply get the time derivative of the total angular momentum for the charges and currents. In summary, we have

$$\frac{\mathrm{d}\mathbf{L}_{\mathrm{mech}}}{\mathrm{d}t} = \int \mathrm{d}^3x \left[\mathbf{r}\times\left(\rho\mathbf{E} + \tfrac{1}{c}\mathbf{J}\times\mathbf{B}\right)\right].$$

As for the first term, $\mathbf{S}$ is related to the momentum of the magnetic field since if we consider waves propagating, it points in the direction of propagation, so we can consider $\mathbf{r}\times\mathbf{S}/c^2$ to be the angular momentum *density* of the electromagnetic field, and upon integrating it over the volume we have

$$\frac{\mathrm{d}\mathbf{L}_{\mathrm{em}}}{\mathrm{d}t} = \frac{1}{c^2}\int \mathrm{d}^3x \frac{\partial}{\partial t}\left(\mathbf{r}\times\mathbf{S}\right).$$

Thus, we have shown that Eq. (2), when integrated over all space corresponds to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{L}_{\mathrm{em}} + \mathbf{L}_{\mathrm{mech}}\right] = \oint_S \hat{\mathbf{x}}_n \epsilon^{nlm} x^l T^{(M)}_{mk} n_k \, \mathrm{d}A.$$

This can be analyzed in much the same way as the equivalent for momentum. The left is obviously the rate of change of the total angular momentum and the term in the integral on the right is the torque per unit area flowing across the surface S.

Problem 2. Stress-tensor in action

Consider two parallel, infinite charged planes with (surface) charge density equal to σ and $-\sigma$.

a. Calculate the electric field generated by them.

b. Calculate the potential generated by them.

c. Using the result in b), calculate the force (per unit area) between the planes.

d. Calculate the stress tensor.

e. Using the conservation of momentum law, calculate the force between the planes by integrating the stress tensor over a surface separating the two planes.

Solution

Part a.

First consider one infinite plane of charge σ and we draw a Gaussian surface over it shown (with variables defined) in Fig. 1. It is the a straightforward calculation:

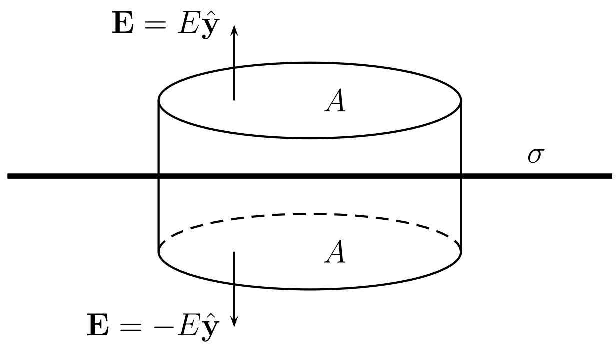

Figure 1: The Gaussian surface about our infinite plane of charge. Symmetry tells us that the electric takes the form shown in the figure.

$$\int \mathbf{E} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = 4\pi Q_{\text{enc}}$$
$$2EA = 4\pi\sigma A,$$

from which we have:

$$\mathbf{E}_{\text{one plate}} = \begin{cases} 2\pi\sigma\hat{\mathbf{y}}, & y > 0, \\ -2\pi\sigma\hat{\mathbf{y}}, & y < 0, \end{cases}$$

for one plane of charge. Putting two together just requires s-s-superposition. And we see that if they are equal and opposite, we get zero field outside and double fiend inside, so in between the plates we have (assume the σ sheet is at a lower y coordinate than the $-\sigma$ sheet):

$$\mathbf{E} = 4\pi\sigma\hat{\mathbf{y}}. \tag{3}$$

Part b.

A perfectly reasonable way to get the potential is to just guess its form, and we need something such that

$$-\boldsymbol{\nabla}\phi = 4\pi\sigma\hat{\mathbf{y}},$$

so we can already see it's independent of x and z, and it must give a constant in the $\hat{\mathbf{y}}$ direction. Thus, we can simply make it:

$$\phi(x, y, z) = -4\pi\sigma y. \tag{4}$$

Part c.

To find the force per unit area, we can appeal to the energy density (with $\mathbf{B} = 0$)

$$u = \frac{1}{8\pi}(\boldsymbol{\nabla}\phi)^2 = 2\pi\sigma^2,$$

and if integrate it over space we get the energy

$$U = \int_0^L \mathrm{d}y \int \mathrm{d}^2x\, 2\pi\sigma^2,$$

then the force is given by

$$\frac{\mathrm{d}U}{\mathrm{d}L} = \int \mathrm{d}^2x\, 2\pi\sigma^2.$$

If the plates have area A then we can find the force per unit area

$$\frac{F}{A} = -\frac{1}{A}\frac{\mathrm{d}U}{\mathrm{d}L} = -2\pi\sigma^2,$$

the force is attractive, as one would expect.

Part d.

To find the stress tensor we have the general form:

$$T_{ij}^{(M)} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right],$$

Since we only have a y component of $\mathbf{E}$ and no $\mathbf{B}$, we will have a purely diagonal $T^{(M)}$, so we get

$$T_{xx}^{(M)} = T_{zz}^{(M)} = -\frac{1}{8\pi}E^2 = -2\pi\sigma^2,$$
$$T_{yy}^{(M)} = \frac{1}{8\pi}E^2 = 2\pi\sigma^2.$$

Part e.

Now, to find the force between them we have the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{em}}\right]_i = \oint_S T_{ij}^{(M)} n_j \, \mathrm{d}A,$$

and we can complete the surface around one of the plates (the lower one, so that the normal is just $\hat{\mathbf{y}}$ in between the plates), but since the field is zero outside the only contribution will be from the portion directly in between which we take to be a flat plane constant in y. Thus, we get (since $\mathbf{P}_{\text{em}} = 0$),

$$F_i = \int T_{iy}^{(M)} \mathrm{d}^2 x,$$

where $\mathrm{d}^2 x$ is over x and z and so we have the only non-zero component of the force being

$$F_y = 2\pi\sigma^2 A,$$

but we need to be careful with the intepretation here. In part c. we got a negative sign which indicated an attractive force (L got smaller). In this problem, we have drawn our surface around the lower plate (the positively charged one), so this indicates that the lower plate has a force of $2\pi\sigma^2 A$ pulling it up, which decreases the distance between the plates, just as expected.

NOTE: One might ask the question: What if the charge densities aren't equal and opposite? In this case the force per unit area on one plate is easily found to be $F/A = -2\pi\sigma_1\sigma_2$ (let's say one has charge σ_1 and the other $-\sigma_2$). On the otherhand, the total electric field *between* the plates is $2\pi(\sigma_1 + \sigma_2)$, so the energy density between the plates is

$$u = \frac{\pi}{2}\left(\sigma_1 + \sigma_2\right)^2,$$

and blindly going through the same procedure as before we get

$$\frac{F}{A} = -\frac{\pi}{2}\left(\sigma_1 + \sigma_2\right)^2, \qquad \text{(WRONG)}.$$

This is clearly not the same as the force we can calculate between the two plates, so what is missing? Well, we neglected the fact that when the plates move closer together, the energy density *outside* the plates changes as well (a point not needed if the charge densities are equal since it is always zero outside). The total energy is infinite though, so we need a way to get a finite force from it, so we call the energy zero when the plates are a distance L_0 apart. Considering this, outside the electric field has a magnitude $2\pi(\sigma_1 - \sigma_2)$, so when the plates are a distance L apart, we have the total energy (per unit area):

$$\frac{U}{A} = \frac{\pi}{2}\left(\sigma_1 + \sigma_2\right)^2 (L - L_0) + \frac{\pi}{2}\left(\sigma_1 - \sigma_2\right)^2 (L_0 - L),$$

and we can now differentiate to get the force per unit area:

$$\frac{F}{A} = -\frac{\pi}{2}\left[(\sigma_1 + \sigma_2)^2 - (\sigma_1 - \sigma_2)^2\right] | = -2\pi\sigma_1\sigma_2,$$

the result we got before! The key was recognizing that the energy density *outside* the plates changed.

Such a procedure is important in finding things like the Casimir effect (where there is a force due to quantum fluctuations in the electromagnetic field which causes two parallel, conducting, electrically neutral plates to attract). In the Casimir effect, if the QED vacuum energy outside of the plates is not included, the results one gets are not only nonsense, they are infinite! Including the energy density outside the plates, one gets a nice finite force which has been experimentally observed.

Problem 3. Multipole expansion

Consider a linear distribution of charge along the z-axis where the charge density from $z = 0$ to $z = L/2$ is λ and from $z = 0$ to $z = -L/2$ is $-\lambda$.

a. Calculate the potential by direct integration.

b. Calculate the total charge, dipole moment and quadrupole moment of the charge distribution.

c. Calculate the potential using the multipole expansion up to third (quadrupole) order.

d. Compare the results in a) and c) and verify that they agree to the order they should agree.

Solution

The charge density for this problem is given by

$$\rho(\mathbf{x}) = \begin{cases} 0, & z > L/2, \\ \lambda\,\delta(x)\delta(y), & 0 < z < L/2, \\ -\lambda\,\delta(x)\delta(y), & -L/2 < z < 0, \\ 0, & z < -L/2. \end{cases}$$

The potential can be found then

$$\begin{aligned} \phi(\mathbf{x}) &= \int \mathrm{d}^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &= \lambda\left[\int_0^{L/2} \frac{\mathrm{d}z'}{|\mathbf{x}-z'\hat{\mathbf{z}}|} - \int_{-L/2}^{0} \frac{\mathrm{d}z'}{|\mathbf{x}-z'\hat{\mathbf{z}}|}\right] \\ &= \lambda\left[\int_0^{L/2} \frac{\mathrm{d}z'}{|\mathbf{x}-z'\hat{\mathbf{z}}|} - \int_0^{L/2} \frac{\mathrm{d}z'}{|\mathbf{x}+z'\hat{\mathbf{z}}|}\right]. \end{aligned}$$

We can evaluate the integral as follows:

$$\begin{aligned} \int_0^{L/2} \frac{\mathrm{d}z'}{|\mathbf{x}-z'\hat{\mathbf{z}}|} &= \int_0^{L/2} \frac{\mathrm{d}z'}{\sqrt{r^2+z'^2-2rz'\cos\theta}} \\ &= \int_0^{L/2} \frac{\mathrm{d}z'}{\sqrt{r^2\sin^2\theta+(z'-r\cos\theta)^2}} \\ &= \int_{-\operatorname{arcsinh}(\cot\theta)}^{\operatorname{arcsinh}(L/2r\sin\theta-\cot\theta)} \frac{\cosh\nu\,\mathrm{d}\nu}{\sqrt{\sinh^2\nu+1}}, \qquad z'-r\cos\theta = r\sin\theta\sinh\nu, \\ &= \int_{-\operatorname{arcsinh}(\cot\theta)}^{\operatorname{arcsinh}(L/2r\sin\theta-\cot\theta)} \mathrm{d}\nu \\ &= \operatorname{arcsinh}\left[\frac{L}{2r\sin\theta}-\cot\theta\right] + \operatorname{arcsinh}(\cot\theta). \end{aligned}$$

Similarly,

$$\int_0^{L/2} \frac{\mathrm{d}z'}{|\mathbf{x}+z'\hat{\mathbf{z}}|} = \operatorname{arcsinh}\left[\frac{L}{2r\sin\theta}+\cot\theta\right] - \operatorname{arcsinh}(\cot\theta).$$

Thus, we have for the potential

$$\phi(\mathbf{x}) = \lambda\left[2\operatorname{arcsinh}(\cot\theta) - \operatorname{arcsinh}\left(\cot\theta+\frac{L}{2r\sin\theta}\right) - \operatorname{arcsinh}\left(\cot\theta-\frac{L}{2r\sin\theta}\right)\right]. \quad (5)$$

This can be put in terms of logarithms and such, but that will just complicate matters. Besides, the arcsinh function is a well-defined function, unlike arcsin which needs its range carefully defined (and indeed, unlike log since log is not defined for negative numbers without appeal to complex analysis).

Part b.

The multipole expansion up to third order takes the form

$$\phi(\mathbf{x}) = \frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\frac{x_i Q_{ij} x_j}{r^5} + O(1/r^4), \quad (6)$$

where we have the moments given by

$$\begin{aligned}\mathbf{p} &= \int \mathbf{x}'\rho(\mathbf{x}')\mathrm{d}^3x' \\ &= 2\lambda\int_0^{L/2} z\hat{\mathbf{z}}\,\mathrm{d}z \\ \mathbf{p} &= \frac{1}{4}\lambda L^2\hat{\mathbf{z}},\end{aligned}$$

and

$$\begin{aligned}Q_{ij} &= \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\mathrm{d}^3x' \\ &= \int \mathrm{d}z(3z'^2\delta_{iz}\delta_{jz} - z'^2\delta_{ij})\lambda(z').\end{aligned}$$

In the expression for Q_{ij} above, we see that we have a line integral of an even function times an odd function ($\lambda(z')$ is odd), so we have

$$Q_{ij} = 0.$$

Thus, we have the expansion for the potential (note that $q = 0$, total charge is zero):

$$\phi(\mathbf{x}) = \frac{\lambda L^2 z}{4r^3} + O(1/r^4),$$

or if we plug in spherical coordinates

$$\phi(\mathbf{x}) = \frac{\lambda L^2\cos\theta}{4r^2} + O(1/r^4). \quad (7)$$

Part c.

We can do this expansion without appeal to logarithms. Remember from calculus way back in the day that we have the property for inverse functions

$$\frac{\mathrm{d}}{\mathrm{d}y}\,\mathrm{arcsinh}(y) = \left.\frac{1}{\mathrm{d}(\sinh x)/\mathrm{d}x}\right|_{x=\mathrm{arcsinh}(y)} = \frac{1}{\cosh[\mathrm{arcsinh}(y)]},$$

so we can find the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\,\mathrm{arcsinh}(y) = -\frac{y}{\cosh^3[\mathrm{arcsinh}(y)]}.$$

Thus, we have the Taylor expansion

$$\mathrm{arcsinh}(y+\epsilon) = \mathrm{arcsinh}(y) + \frac{\epsilon}{\cosh[\mathrm{arcsinh}(y)]} - \frac{y\,\epsilon^2}{2\cosh^3[\mathrm{arcsinh}(y)]} + A(y)\epsilon^3 + O(\epsilon^4),$$

and from this it is clear that

$$\mathrm{arcsinh}(y+\epsilon) + \mathrm{arcsinh}(y-\epsilon) = 2\,\mathrm{arcsinh}(y) - \frac{y\,\epsilon^2}{\cosh^3[\mathrm{arcsinh}(y)]} + O(\epsilon^4).$$

Now, we need to put in that $y = \cot\theta$ and $\epsilon = L/2r\sin\theta$. To see what $\cosh[\mathrm{arcsinh}(y)]$ is we observe that (letting $x = \mathrm{arcsinh}(y)$)

$$\cosh^2 x = \sinh^2 x + 1 = y^2 + 1,$$

and when $y = \cot^2\theta$, we have

$$\cosh^2 x = 1/\sin^2\theta,$$

and since θ only ranges from 0 to π, we can take the square root and keep the positive root. Thus, if we plug in the appropriate stuff the expansion becomes

$$\begin{aligned}\mathrm{arcsinh}(y+\epsilon) + \mathrm{arcsinh}(y-\epsilon) &= 2\,\mathrm{arcsinh}(\cot\theta) - \cot\theta\sin^3\theta\left(\frac{L}{2r\sin\theta}\right)^2 + O(\epsilon^4)\\ &= 2\,\mathrm{arcsinh}(\cot\theta) - \cos\theta\frac{L^2}{4r^2} + O(L/r)^4.\end{aligned}$$

If we take this expansion and plug into Eq. (5) we obtain

$$\phi(\mathbf{x}) = \frac{\lambda L^2\cos\theta}{4r^2} + O(L/r)^4. \tag{8}$$

Part d.

Observe that Eq. (7) and Eq. (8) are equal up to the required order.

Problem 4. Most of vector analysis

The components notation ("indices galore") and tensors are also useful in 3 dimensions. The trick is to write the vector product with the help of the 3-D ϵ^{ijk} tensor as

$$(\mathbf{A} \times \mathbf{B})^i = \epsilon^{ijk} A^j B^k.$$

a. Show that $\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$ (hint: what else could it be?)

b. Show that $\epsilon^{ijk}\epsilon^{ijm} = 2\delta^{km}$.

c. Show that

$$\begin{aligned}
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \\
\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) &= \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} \\
\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B} \\
\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{A} &= 0 \\
\boldsymbol{\nabla} \times \boldsymbol{\nabla} f &= 0 \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A},
\end{aligned}$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are vector fields and f a scalar field (all well behaved enough).

Solution

Part a.

This clearly only needs to be checked when $j \neq k$ and $l \neq m$ since otherwise, it is clear that it works. Also, we need either $j = l$ and $k = m$ or $j = m$ and $k = l$ since otherwise the sum over i will never produce a non-zero result, and by anti-symmetry, these two cases are the same. So we have only one case to test with $a \neq b$ (no sum over a and b in what is to follow, furthermore let c be the only remaining component not equal to a or b)

$$\epsilon^{iab}\epsilon^{iab} = (\epsilon^{cab})^2 = 1 = \delta^{aa}\delta^{bb} - \delta^{ab}\delta^{ab}.$$

Thus, we have established

$$\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}. \tag{9}$$

Part b.

This sum is a simple application of Eq. (9):

$$\epsilon^{ijk}\epsilon^{ijm} = \delta^{jj}\delta^{km} - \delta^{jm}\delta^{kj} = 3\delta^{km} - \delta^{km} = 2\delta^{km}. \tag{10}$$

Part c.

The following vector identities are all proven with index notation. The only thing difficult is wrapping one's head around relating a given vector identity to it's component analog and how to algebraically handle the indices.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A^i\epsilon^{ijk}B^jB^k = \epsilon^{ijk}A^iB^jC^k. \tag{11}$$

We can see from Eq. (11) that we can cyclically permute A, B, and C and with some relabeling of dummy indices, obtain the same expression (this in fact justifies using determinants to evaluate cross products, since Eq. (11) is just the determinant of the matrix with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as rows), so we have

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}).$$

Moving along,

$$\begin{aligned}
[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})]^i &= \epsilon^{ijk}A^j\epsilon^{klm}\partial_l B^m + \epsilon^{ijk}B^j\epsilon^{klm}\partial_l A^m \\
&= \epsilon^{kij}\epsilon^{klm}\left(A^j\partial_l B^m + B^j\partial_l A^m\right) \\
&= \left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)\left(A^j\partial_l B^m + B^j\partial_l A^m\right) \\
&= A^j\partial_i B^j - A^j\partial_j B^i + B^j\partial_i A^j - B^j\partial_j A^i \\
&= \partial_i(A^jB^j) - A^j\partial_j B^i - B^j\partial_j A^i \\
&= [\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}]^i.
\end{aligned}$$

So re-arranging things we have,

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}).$$

Note that this rule tells us that the $BAC-CAB$ rule *does not* work when there is a gradient in the mix (unless there are two gradients, and then because they commute with each other).

Moving along,

$$\begin{aligned}
\boldsymbol{\nabla}(\mathbf{A}\times\mathbf{B}) &= \partial_i\left(\epsilon^{ijk}A^jB^k\right) \\
&= \epsilon^{ijk}(\partial_i A^j)B^k + \epsilon^{ijk}A^j(\partial_i B^k) \\
&= B^k\epsilon^{kij}\partial_i A^j - A^j\epsilon^{jik}\partial_i B^k \\
&= \mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{B}. \\
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} &= \partial_i\epsilon^{ijk}\partial_j A^k \\
&= \epsilon^{ijk}\partial_j\partial_i A^k \\
&= -\partial_j\epsilon^{jik}\partial_i A^k \\
&= -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} \\
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} &= 0 \\
\boldsymbol{\nabla}\times\boldsymbol{\nabla}f &= \epsilon^{ijk}\partial_j\partial_k f
\end{aligned}$$

$$
\begin{aligned}
&= \epsilon^{ijk}\partial_k\partial_j f \\
&= -\epsilon^{ikj}\partial_k\partial_j f \\
&= -\boldsymbol{\nabla}\times\boldsymbol{\nabla} f \\
\boldsymbol{\nabla}\times\boldsymbol{\nabla} f &= 0.
\end{aligned}
$$

And finally, we have

$$
\begin{aligned}
[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})]^i &= \epsilon^{ijk}\partial_j\epsilon^{klm}\partial_l A^m \\
&= \epsilon^{kij}\epsilon^{klm}\partial_j\partial_l A^m \\
&= (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\partial_j\partial_l A^m \\
&= \partial_i\partial_j A^j - \partial_j\partial_j A^i \\
&= [\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}]^i.
\end{aligned}
$$

And we thus have,

$$
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}.
$$

Problem 5. Be creative

a. Invent a problem on the stuff we are studying in class, or related to it. Prizes will be given for fanciness and originality.

b. Solve the problem in a).

Solution

The problem should *not* be trivial (like finding the electric field of a point charge, simple Gaussian surface problems, etc.). In theory, it should not be copied out of some book like Jackson or something, but I can't check that so I'll just trust you guys.

Part a.

Grade your homework. I know, not very creative, but they don't pay me enough for that (as you know `:-P`).

Part b.

This'll be solved when you get your homework back.