Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 10.1 Radiation from Spinning Electric Dipole

An electric dipole of size p_0 lying on the x-y plane rotates around the z-axis with angular velocity ω. Calculate the power radiated per solid angle and the total radiated power.

Solution

We can write a rotating dipole as

$$\mathbf{p} = p_0[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}] = \Re\left\{p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega t}\right\}.$$

This form of the dipole moment is what is used in Jackson 9.1 and 9.2 to derive the form of the vector potential (9.16), so we have the vector potential

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= -\frac{i\omega}{c}\mathbf{p}\frac{e^{ikr}}{r} \\ &= -\frac{i\omega p_0}{c}\frac{e^{ikr}}{r}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}).\end{aligned}$$

In the radiation zone we get the fields:

$$\begin{aligned}\mathbf{H} &= k^2(\hat{\mathbf{r}} \times \mathbf{p})\frac{e^{ikr}}{r}, \\ \mathbf{E} &= \mathbf{H} \times \hat{\mathbf{r}},\end{aligned}$$

and so we get that the power radiated per solid angle is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c}{8\pi}\Re\left[r^2\hat{\mathbf{r}} \cdot \mathbf{E} \times \mathbf{H}^\star\right],$$

and we have

$$\begin{aligned}r^2\hat{\mathbf{r}} \cdot \mathbf{E} \times \mathbf{H}^\star &= r^2\hat{\mathbf{r}} \cdot (\mathbf{H} \times \hat{\mathbf{r}}) \times \mathbf{H}^\star \\ &= k^4\hat{\mathbf{r}} \cdot [(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}}] \times (\hat{\mathbf{r}} \times \mathbf{p}^\star) \\ &= k^4\left(\hat{\mathbf{r}} \times \mathbf{p}^\star\right) \cdot \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times (\mathbf{p} \times \hat{\mathbf{r}})] \\ &= k^4\left(\hat{\mathbf{r}} \times \mathbf{p}^\star\right) \cdot \hat{\mathbf{r}} \times [\mathbf{p} - \hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}})] \\ &= k^4\left(\hat{\mathbf{r}} \times \mathbf{p}^\star\right) \cdot \left(\hat{\mathbf{r}} \times \mathbf{p}\right).\end{aligned}$$

We can further write

$$\hat{\mathbf{x}} + i\hat{\mathbf{y}} = e^{i\phi}\left[\sin\theta\,\hat{\mathbf{r}} + i\hat{\boldsymbol{\phi}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right],$$

so that

$$\hat{\mathbf{r}} \times \mathbf{p} = p_0 e^{i\phi}\left[i\hat{\boldsymbol{\theta}} - \cos\theta\,\hat{\boldsymbol{\phi}}\right].$$

Thus, we have

$$r^2\hat{\mathbf{r}} \cdot \mathbf{E} \times \mathbf{H}^\star = k^4 p_0^2(1 + \cos^2\theta).$$

The power radiated by this dipole is then

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{ck^4}{8\pi} p_0^2 (1 + \cos^2\theta).$$

Note: This power is the same as the addition of power radiated per solid angle for an oscillating dipole in the x-direction and an oscillating dipole in the y-direction (out of phase by 90 degrees).

The total power is then

$$P = \frac{2}{3} ck^4 p_0^2.$$

Problem 10.2 Classical Atoms don't Exist

In a classical model of atoms, the electron spins around the nucleus in a circular orbit. Classically, it should radiate, lose energy and eventually fall into the nucleus.

a. Calculate how much energy the electron radiates per turn.

b. Estimate how long it'd take for the electron in the ground state of a hydrogen atom to hit the nucleus (order of magnitude only).

Solution

For this we use the Lorentz invariant Larmor formula for circular motion (see (14.26) in Jackson or (14.47) in Jackson):

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left|\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right|^2.$$

The last term is the force on the electron, and in the case of an electron

$$\left|\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right| = \frac{e^2}{R^2},$$

where R is the radius to the nucleus. Now, simple mechanics arguments give the angular velocity for a circular orbit

$$\omega^2 = \frac{e^2}{mR^3},$$

so the period of motion is

$$T = \frac{2\pi R^{3/2}m^{1/2}}{e}.$$

Thus, in one period we have radiated this much energy away:

$$E = \frac{4\pi}{3}\frac{e^5}{m^{3/2}c^3R^{5/2}}\gamma^2.$$

For the non-relativistic limit $\gamma = 1$. Now, going back to power radiated we have

$$P = \frac{2}{3}\frac{e^6}{m^2c^3R^4}.$$

The total energy the electron has is

$$U = -\frac{e}{2R},$$

The time derivative of this is given as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{e^2\dot{R}}{2R^2}.$$

Thus, if we equate this change of energy with power radiated, we get

$$R^2\dot{R} = -\frac{4}{3}\frac{e^4}{m^2c^3} = \frac{1}{3}\frac{\mathrm{d}R^3}{\mathrm{d}t},$$

so that we get

$$R^3 = R_0^3 - \frac{4e^4t}{m^2c^3},$$

so we get a collapse of the electron to the nucleus in

$$t = \frac{c^3R_0^3m^2}{4e^4}.$$

We have $R_0 \sim 10^{-11}\,\mathrm{m}$ and $e^2/mc^2 \sim 10^{-15}\,\mathrm{m}$. Thus, we have

$$t \sim 10^{-8} \cdot 10^{-33} \cdot 10^{30}\,\mathrm{s} \sim 10^{-11}\,\mathrm{s}.$$

Thus, in 10^{-11} seconds the electron should fall into the nucleus due to radiation.