Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 1. Field invariants

a. Write the Lorentz scalars $F_{\mu\nu}F^{\mu\nu}$ and $\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

b. Show that if $\mathbf{E}$ and $\mathbf{B}$ are seen as perpendicular in one reference frame, they are perpendicular in any other reference frame.

c. Show that if $|\mathbf{E}| = |\mathbf{B}|$ in one reference frame, $|\mathbf{E}'| = |\mathbf{B}'|$ in any other reference frame.

d. Show that if $|\mathbf{E}| > |\mathbf{B}|$ in one reference frame, $|\mathbf{E}'| > |\mathbf{B}'|$ in any other reference frame.

Solution

Part a.

First, what is a good way to remember the form of $F_{\mu\nu}$? We can write out $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, but we need to remember what the various components of A_ν are:

$$A^\mu = (\Phi, \mathbf{A}) \qquad \Longrightarrow \qquad A_\mu = (\Phi, -\mathbf{A}).$$

Where Φ is the potential and $\mathbf{A}$ is the vector potential. In the rest of the problem, when referring to components of $\mathbf{A}$, a lower case 'a' will be used to differentiate between A_μ (i.e. a_i is the ith component of $\mathbf{A}$). On top of this, we have the fact that

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_0\mathbf{A},$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.$$

Using the relations between $\mathbf{E}$, $\mathbf{B}$ and the potentials, we can see that

$$F_{0i} = \partial_0 A_i - \partial_i\Phi = -\partial_0 a_i - \partial_i\Phi = E_i,$$
$$F_{ij} = \partial_i A_j - \partial_j A_i = \partial_j a_i - \partial_i a_j.$$

To see how the F_{ij} are related to $\mathbf{B}$, we consider

$$\begin{aligned}\epsilon_{ijk}B_k &= \epsilon_{kij}\epsilon_{klm}\partial_l a_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_l a_m \\ &= \partial_i a_j - \partial_j a_i \\ &= -F_{ij}.\end{aligned}$$

Thus, we have shown that the components of $F_{\mu\nu}$ are

$$F_{0i} = E_i, \tag{1}$$
$$F_{ij} = -\epsilon_{ijk}B_k. \tag{2}$$

Eq. (1) and Eq. (2) will be useful in evaluating the scalars above.

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} &= F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= \epsilon_{ijk}\epsilon_{ijl}B_k B_l - 2E_i E_i \\ &= 2\delta_{kl}B_k B_l - 2E_i E_i \\ &= 2(B^2 - E^2).\end{aligned}$$

For the last scalar, note that if one index of the completely antisymmetric tensor $\epsilon^{\mu\nu\lambda\rho}$ is 0, then the others must be spatial, and by convention $\epsilon_{0123} = 1$, so $\epsilon^{0123} = -1$. Thus, we have

$$\begin{aligned}
\epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} &= \epsilon^{0ijk} F_{0i} F_{jk} + \epsilon^{i0jk} F_{i0} F_{jk} + \epsilon^{jk0i} F_{jk} F_{0i} + \epsilon^{jki0} F_{jk} F_{i0} \\
&= 4\epsilon^{0ijk} F_{0i} F_{jk} \\
&= 4\epsilon_{ijk} E_i \epsilon_{jkl} B_l \\
&= 4\epsilon_{jki} \epsilon_{jkl} E_i B_l \\
&= 8\delta_{il} E_i B_l \\
&= 8\mathbf{E} \cdot \mathbf{B}.
\end{aligned}$$

Thus, we have shown that

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2), \tag{3}$$

$$\epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = 8\mathbf{E} \cdot \mathbf{B}. \tag{4}$$

Part b.

Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ in one reference frame (i.e. $\mathbf{E}$ is perpendicular to $\mathbf{B}$), but we have shown that this is a Lorentz scalar in part a, so if we goto a new frame, we still have $\mathbf{E}' \cdot \mathbf{B}' = 0$.

Part c.

Assume $E = B$ in one frame of reference, but again, in part a. we showed that $B^2 - E^2 = B'^2 - E'^2$ where B' and E' are in a different frame of reference, so if $B = E$, then we have $E' = B'$ in any other reference frame.

Part d.

Using the same relation as in part c, if we assume $E > B$, we have

$$B'^2 - E'^2 = B^2 - E^2 < 0,$$

and thus we trivially have $E' > B'$.

Problem 2. Field of moving charge

What are the **E** and **B** fields of a point particle with charge q moving along the z axis with constant speed v. Sketch the fields.

Solution

We need to Lorentz transform the electromagnetic field of a point charge. To do this, we could take the matrix $F_{\mu\nu}$ and Lorentz transform it, but that would take the multiplication of three matrices. To avoid this, we Lorentz transform the four-potential, which for this case takes the form

$$A'^{\mu} = \left(\frac{q}{r}, \mathbf{0}\right)$$

in the particle's restframe. In order to get the particle going with velocity v in the z direction, we need to do a Lorentz transformation with velocity $-v$ in the z-direction. With the standard definitions

$$\beta = \frac{v}{c}, \tag{5}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{6}$$

we have the Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}. \tag{7}$$

Thus, we have

$$A^{\mu} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} q/r \\ 0 \\ 0 \\ 0 \end{pmatrix} = \gamma q \begin{pmatrix} 1/r \\ 0 \\ 0 \\ \beta/r \end{pmatrix}.$$

Unfortunately, this is not quite the answer since r is evaluated in the moving frame, and we must Lorentz transform each coordinate. All directions orthogonal to motion will remain unchanged, but we will have $z' = \gamma(z - \beta ct)$, so we can read off directly what the potentials for this case are:

$$\Phi = \frac{\gamma q}{\sqrt{x^2 + y^2 + \gamma^2(z-\beta ct)^2}}, \tag{8}$$

$$\mathbf{A} = \frac{\gamma\beta q}{\sqrt{x^2 + y^2 + \gamma^2(z-\beta ct)^2}}\hat{\mathbf{z}}. \tag{9}$$

To find the fields, we need to take the following derivatives

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-\beta ct)^2}} = -\frac{x}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-\beta ct)^2}} = -\frac{y}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$\frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-\beta ct)^2}} = -\frac{\gamma^2(z-\beta ct)}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$\frac{1}{c}\frac{\partial}{\partial t}\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-\beta ct)^2}} = \frac{\gamma^2\beta(z-\beta ct)}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}}.$$

This leads to the electric field (from $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_0\mathbf{A}$)

$$E_x = \frac{\gamma q x}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$E_y = \frac{\gamma q y}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$E_z = \frac{\gamma q(z-vt)}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}}.$$

And from $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, we have that there are only x and y components of $\mathbf{B}$. These components take the form $B_x = \partial_y A_z$ and $B_y = -\partial_x A_z$, and so we have

$$B_x = -\frac{\gamma\beta q y}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$B_y = \frac{\gamma\beta q x}{\left(x^2+y^2+\gamma^2(z-\beta ct)^2\right)^{3/2}},$$
$$B_z = 0.$$

To plot, we can take $t = 0$, and recast this in the language of three-vectors using $\psi = \cos^{-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})$ (where $\hat{\mathbf{n}}$ is the unit vector pointing from the particle to the point of observation)

$$\mathbf{E} = \frac{\gamma q\mathbf{r}}{\gamma^3 r^3[\gamma^{-2}(x^2+y^2)/r^2 + z^2/r^2]^{3/2}}$$
$$= \frac{q\mathbf{r}}{\gamma^2 r^3[1-\beta^2(x^2+y^2)/r^2]^{3/2}}$$
$$\mathbf{E} = \frac{q\mathbf{r}}{\gamma^2 r^3[1-\beta^2\sin^2\psi]^{3/2}}.$$

Similarly, we have

$$\mathbf{B} = \frac{q\,\mathbf{r}\times\mathbf{v}}{c\gamma^2 r^3[1-\beta^2\sin^2\psi]^{3/2}}.$$

Using these, we notice that we get stronger field lines when $\psi = \pi/2$ for both $\mathbf{E}$ and $\mathbf{B}$, and that $\mathbf{B}$ circles about the z-axis.

Problem 3. Tensor (anti-)symmetry

Show that if $A_{\mu\nu} = A_{\nu\mu}$ is a symmetric tensor in one frame, it'll be symmetric in any other frame. Is *anti*-symmetry ($A_{\mu\nu} = -A_{\nu\mu}$) also frame independent?

Solution

Assume $A_{\mu\nu}$ is symmetric. The Lorentz transformation then takes the form

$$A'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta A_{\alpha\beta} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta A_{\beta\alpha} = \Lambda_\nu{}^\beta \Lambda_\mu{}^\alpha A_{\beta\alpha} = \Lambda_\nu{}^\alpha \Lambda_\mu{}^\beta A_{\alpha\beta} = A'_{\nu\mu}. \tag{10}$$

On the otherhand, if $A_{\mu\nu}$ is antisymmetric

$$A'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta A_{\alpha\beta} = -\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta A_{\beta\alpha} = -\Lambda_\nu{}^\beta \Lambda_\mu{}^\alpha A_{\beta\alpha} = -\Lambda_\nu{}^\alpha \Lambda_\mu{}^\beta A_{\alpha\beta} = -A'_{\nu\mu}. \tag{11}$$

Thus, if A is symmetric in one frame, it is symmetric in all frames, and the same goes for if A is antisymmetric.

Problem 4. Relativistic velocity composition

Multiply two Lorentz transformation matrices with velocities v_1 and v_2 (you can take v_1 parallel to v_2) and show that the result also looks like a Lorentz transformation with velocity v_3. What is v_3 as a function of v_1 and v_2?

Solution

If we take v_1 parallel to v_2 for two Lorentz Transformations, we can isolate the two-by-two matrix of time and the direction of the velocities, so that we have for our two transformations

$$\begin{aligned}
\Lambda^\mu{}_\nu &= \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \\
\Lambda'^\mu{}_\nu &= \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \\
\Lambda''^\mu{}_\nu &= \Lambda'^\mu{}_\alpha \Lambda^\alpha{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}.
\end{aligned} \tag{12}$$

From this point, we multiply matrices, so we get

$$\begin{aligned}
\Lambda''^\mu{}_\nu &= \gamma_1\gamma_2 \begin{pmatrix} 1 & -\beta_2 \\ -\beta_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta_1 \\ -\beta_1 & 1 \end{pmatrix} \\
&= \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_2 & -\beta_1-\beta_2 \\ -\beta_1-\beta_2 & 1+\beta_1\beta_2 \end{pmatrix}.
\end{aligned} \tag{13}$$

Now, relate Eq. (12) to Eq. (13), and we get from the first component:

$$\begin{aligned}
\gamma &= \gamma_1\gamma_2(1+\beta_1\beta_2) \\
\frac{1}{1-\beta^2} &= \frac{(1+\beta_1\beta_2)^2}{(1-\beta_1^2)(1-\beta_2)^2} \\
1-\beta^2 &= \frac{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2}{1+2\beta_1\beta_2+\beta_1^2\beta_2^2} \\
\beta^2 &= 1-\frac{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2}{1+2\beta_1\beta_2+\beta_1^2\beta_2^2} \\
\beta^2 &= \frac{\beta_1^2+2\beta_1\beta_2+\beta_2^2}{1+2\beta_1\beta_2+\beta_1^2\beta_2^2} \\
\beta &= \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \\
v_3 &= \frac{v_1+v_2}{1+v_1v_2/c^2}.
\end{aligned} \tag{14}$$

This is the velocity of the Lorentz transformation that comes from first boosting by v_1 then by v_2, and the resulting transformation is indeed a Lorentz transformation.

Problem 5. Relativistic invariance of Maxwell's equations on your face

An infinitely long straight wire of negligible cross sectional area is at rest and has a uniform charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with velocity v parallel to the direction of the wire with respect to the lab frame K.

a. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the lab.

b. What are the charges and current densities associated with the wire in its rest frame? In the lab frame?

c. From the lab frame charge and current densities, calculate directly the electric and magnetic fields in the lab frame. Compare with part a.

Solution

Part a.

We place our z'-axis on top of the wire, and in the K' frame we can put a Gaussian cylinder of radius $r' = x'^2 + y'^2$ and length l' around the wire to evaluate Gauss's law:

$$\begin{aligned}\int \mathbf{E}' \cdot \hat{\mathbf{n}} \mathrm{d}A &= 4\pi Q_{\text{enc}} \\ E'_r(2\pi r' l') &= 4\pi q_0 l' \\ E'_r &= \frac{2q_0}{r'}.\end{aligned}$$

This field is derivable from the potential

$$\Phi' = -2q_0 \log(r'/a),$$

where a is an artificial scale inserted to make the logarithm make sense. In principle, its inclusion is not always needed.

Thus, with no magnetic field, we have our four-potential

$$A'^{\mu} = (-2q_0 \log(r'/a), \mathbf{0}). \tag{15}$$

Before transforming this four-vector, we state what the electric and magnetic fields are in the K' frame

$$\begin{aligned}\mathbf{E}' &= \frac{2q_0}{r'}\hat{\mathbf{r}}', \\ \mathbf{B}' &= \mathbf{0}.\end{aligned}$$

Now, we need to transform A' by a Lorentz transformation with velocity $-v$ along wth z-axis. Thus, we have

$$A^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -2q_0 \log(r'/a) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= -2\gamma q_0 \log(r'/a) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \beta \end{pmatrix}$$

Just as in problem 2, r' is evaluated in the moving frame K', so we must transform it back. And again, the only coordinate effected is $z' = \gamma(z - \beta ct)$, but r' is in cylindrical coordinates, so $r' = r$! Thus, we can easily read off the electric field since there is no time dependence in $\mathbf{A}$ and the potential has the same form as in the K' frame (with modified charge density).

On the other hand, for the magnetic field, we can evaluate

$$\mathbf{B} = \boldsymbol{\nabla} \times \hat{\mathbf{z}} A_z(r) = -(\partial_r A_z)\hat{\boldsymbol{\phi}},$$

where ϕ is the angle around the z-axis. Thus, we have for the electric and magnetic fields

$$\mathbf{E} = \frac{2\gamma q_0}{r}\hat{\mathbf{r}}, \tag{16}$$
$$\mathbf{B} = \frac{2\gamma\beta q_0}{r}\hat{\boldsymbol{\phi}}. \tag{17}$$

Part b.

In the rest frame of the wire, there is no current, so we have the four-current in the K' frame:

$$J'^\mu = (c\rho', \mathbf{J}') = (cq_0\delta(x')\delta(y'), \mathbf{0}).$$

This four-current transforms as a four-vector under Lorentz transformations and thus, so we have in the frame K:

$$J^\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} cq_0\delta(x')\delta(y') \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= \gamma q_0 \delta(x)\delta(y) \begin{pmatrix} c \\ 0 \\ 0 \\ v \end{pmatrix},$$

where we have used the fact that $x' = x$ and $y' = y$. Thus, in the frame K, we see a charge density γq_0 and a current $I = \gamma q_0 v$ going along the z-axis.

Part c.

For the line charge in this reference frame, we can make the same type of Gaussian surface as before to obtain

$$\mathbf{E} = \frac{2\gamma q_0}{r}\hat{\mathbf{r}}, \tag{18}$$

and for the magnetic field, we can apply Ampère's law to a circle around the z-axis (of radius r) to obtain

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{s} = 4\pi I_{\mathrm{enc}}/c$$
$$2\pi r B_\phi = 4\pi\gamma q_0 \beta.$$

Solving the above, we have that

$$\mathbf{B} = \frac{2\gamma\beta q_0}{r}\hat{\boldsymbol{\phi}}. \tag{19}$$

Note that Eq. (19) and Eq. (18) completely agree with Eq. (16) and Eq. (17)!