**Notation/Convention**:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

## Problem 1. Cylinder Boundary Problem

Two halves of a long, hollow cylinder of radius $b$ are separated by small, lengthwise gaps, are kept at potentials $V_1$ and $V_2$. Show that the potential inside the cylinder can be written as

$$\phi = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \arctan\left(\frac{2br\sin\theta}{b^2 - r^2}\right)$$

*Hint:*

$$\sum_{n \text{ odd}} \frac{z^n}{n} = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and $\Im \log Re^{i\theta} = \theta$.

### Solution

For this problem, we first note that $\phi$ is unique up to a constant, so we shift $\phi' = \phi - \frac{V_1+V_2}{2}$ so that we can say the top is at potential $V = \frac{V_1-V_2}{2}$ and the bottom is at $-V = \frac{V_2-V_1}{2}$. Doing this, we now note the anti-symmetry of the problem which sets $\phi' = 0$ on the plane in between the two halves of the cylinders. The picture for the problem we now solve is in Fig. 1. Note that we do not have to consider the $z$-direction since it is a very long cylinder, so that reduces this a two-dimensional problem.

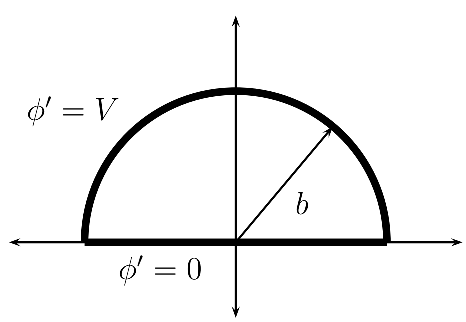


Figure 1: The reduced problem to solve. Note that $V = (V_1 - V_2)/2$.

From this point, it is clear that Laplace's equation for $\phi'$ is

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\phi' = 0. \tag{1}$$

Note that when we separate variables (say $\phi' = R(r)\Theta(\theta)$), we get the boundary conditions:

$$\phi'(\theta = 0) = 0 = \Theta(0),$$
$$\phi'(\theta = \pi) = 0 = \Theta(\pi).$$

Then, we get the two equations

$$\left[r\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} - n^2\right] R(r) = 0, \tag{2}$$
$$-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\Theta(\theta) = n^2\Theta(\theta), \tag{3}$$

and we do not know that $n$ is an integer (yet). With the boundary conditions on $\Theta$, we see that

$$\Theta_n(\theta) = \sin(n\theta), \qquad n = 1, 2, 3, \ldots,$$

As for Eq. (2), we can make the substitution $u = \log(r/b)$, and in that case, we get $r\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}u}$ and thus we have

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}R = n^2 R,$$

and so $R$ is a combination of $e^{nu}$ and $e^{-nu}$ which are equivalent to $(r/b)^n$ and $(b/r)^n$, but we cannot have this blow up at the origin, so we only have $(r/b)^n$ contributing and this gives us the potential

$$\phi'(r,\theta) = \sum_n A_n \left(\frac{r}{b}\right)^n \sin(n\theta).$$

Now, to fit it to a potential $V$ we note:

$$V = \sum_n A_n \sin(n\theta),$$
$$V\int_0^\pi \sin(n\theta) = \frac{\pi}{2}A_n,$$

so that we have

$$A_n = \frac{2V[1-(-1)^n]}{n\pi} = \begin{cases} \frac{4V}{n\pi}, & n \text{ is odd,} \\ 0 & \text{otherwise.}\end{cases}$$

Thus, we have our potential (using the given hints)

$$\begin{aligned}
\phi'(r,\theta) &= \frac{4V}{\pi}\sum_{n\text{ odd}} \frac{\sin(n\theta)}{n}\left(\frac{r}{b}\right)^n \\
&= \frac{4V}{\pi}\Im\sum_{n\text{ odd}}\frac{(re^{i\theta}/b)^n}{n} \\
&= \frac{2V}{\pi}\Im\log\left(\frac{b+re^{i\theta}}{b-re^{i\theta}}\right) \\
&= \frac{2V}{\pi}\Im\log\left(\frac{(b+re^{i\theta})(b-re^{i\theta})}{|b-re^{i\theta}|^2}\right) \\
&= \frac{2V}{\pi}\Im\log\left(\frac{b^2-r^2+2ibr\sin\theta}{|b-re^{i\theta}|^2}\right) \\
&= \frac{2V}{\pi}\arctan\left(\frac{2br\sin\theta}{b^2-r^2}\right).
\end{aligned}$$

Now if we just plug in what $V$ is and add back the constant that we subtracted from the beginning, we have

$$\phi(r,\theta) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \arctan\left(\frac{2br\sin\theta}{b^2 - r^2}\right). \tag{4}$$

## Problem 2. Multipole Fields

a. Calculate the electric field of an electric dipole **P**. *Warning:* there is a delta function potential contribution at the origin, don't miss it. Sketch the field lines.

b. Calculate the electric field of an electric dipole $Q^{ij}$ given by:

$$\mathbf{Q} = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -2Q \end{pmatrix} \tag{5}$$

Sketch the field lines.

### Solution

In the below, we use

$$\partial_i r = \partial_i \sqrt{x_j x_j} = \frac{x_i}{\sqrt{x_j x_j}} = \frac{x_i}{r}$$

### Part a.

The potential for a dipole is given

$$\phi(\mathbf{x}) = \frac{\mathbf{P} \cdot \mathbf{x}}{r^3},$$

and we have the electric field given by $\mathbf{E} = -\boldsymbol{\nabla}\phi$. Then, in components

$$\begin{aligned} E_i = -\partial_i \phi = -\partial_i \left[ \frac{P_j x_j}{r^3} \right] \\ = -\frac{P_i}{(x_k x_k)^{3/2}} + \frac{3P_j x_j x_i}{r^5} \\ = -\frac{P_i x_k x_k - 3P_j x_j x_i}{r^5}. \end{aligned}$$

Going out of components

$$\mathbf{E} = -\frac{\mathbf{P} - 3(\mathbf{P} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}}}{r^3}.$$

### Part b.

The potential for a quadrapole is

$$\phi(\mathbf{x}) = \frac{x_i Q_{ij} x_j}{2r^5},$$

and we have

$$E_i = -\partial_i \left[ \frac{x_j Q_{jk} x_k}{2r^5} \right]$$
$$= -\frac{Q_{ik} x_k}{r^5} + 5 \frac{x_j Q_{jk} x_k x_i}{2r^7}.$$

Now, we find the following:

$$Q_{ik} x_k \hat{\mathbf{x}}_i = Qx\hat{\mathbf{x}} + Qy\hat{\mathbf{y}} - 2Qz\hat{\mathbf{z}} = Qr[(1 - 3\cos^2\theta)\hat{\mathbf{r}} - 3\cos\theta\,\sin\theta\,\hat{\boldsymbol{\theta}}],$$
$$x_j Q_{jk} x_k = Qx^2 + Qy^2 - 2Qz^2 = Qr^2(1 - 3\cos^2\theta),$$

where we have that $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}$. With this, we have

$$\mathbf{E} = -\frac{2Q[(1 - 3\cos^2\theta)\hat{\mathbf{r}} - 3\cos\theta\,\sin\theta\,\hat{\boldsymbol{\theta}}] - 5Q(1 - 3\cos^2\theta)\hat{\mathbf{r}}}{2r^5} \tag{6}$$

$$\mathbf{E} = Q\frac{3}{2}\frac{(1 - 3\cos^2\theta)\hat{\mathbf{r}} + \sin 2\theta\,\hat{\boldsymbol{\theta}}}{r^5}. \tag{7}$$

## Problem 3. Green's function for the Dirichlet problem on a cylinder

A point charge $q$ is located at the point $(\rho', \phi', z')$ inside a grounded cylindrical box defined by the surfaces $z = 0$, $z = L$, and $\rho = a$. Find the potential inside the cylinder. Depending on the method you use, you'll get one of the three forms of the solution found in problem 3.23 in Jackson's book. Can you show the equivalence between them or use different methods to derive more than one form?

### Solution

From the theory of PDE's, if $\psi_n(\mathbf{x})$ are a set of (normalized) eigenfunctions for the Laplacian (i.e., $-\nabla^2\psi_n = k_n^2\psi_n$), then the Green's function takes the form

$$G(\mathbf{x}, \mathbf{x}') = 4\pi q \sum_n \frac{\psi_n^\star(\mathbf{x}')\psi_n(\mathbf{x})}{k_n^2},$$

where $k_n^2$ are the eigenvalues. To see this, apply the Laplacian to get

$$-\nabla^2 G(\mathbf{x}, \mathbf{x}') = 4\pi q \sum_n \frac{\psi_n^\star(\mathbf{x}')(-\nabla^2\psi_n(\mathbf{x}))}{k_n^2} = 4\pi q \sum_n \psi_n(\mathbf{x}')\psi_n(\mathbf{x}) = 4\pi q\, \delta(\mathbf{x} - \mathbf{x}').$$

What are the eigenfunctions then? For these, we need to take into account the boundary conditions: The functions must be well-behaved when $\rho = 0$, go to zero when $\rho = a$, $z = 0$ and $z = L$, and they must be periodic in $\phi$. The Laplacian can be written

$$-\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\psi_n(\mathbf{x}) = k_n^2\psi_n(\mathbf{x}).$$

Separating variables we need

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -K^2 Z \tag{8}$$

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = -\nu^2 Q \tag{9}$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \left(K'^2 - \frac{\nu^2}{\rho^2}\right) R = 0, \tag{10}$$

From the boundary conditions in $Z$ and $Q$ it is easily seen that the normalized eigenfunctions and eigenvalues are

$$Z_k(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi z}{L}\right), \qquad k = 1, 2, 3, \ldots$$

$$Q_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}, \qquad m \in \mathbb{Z}.$$

For Eq. (10) (Bessel's equation), we obviously need to use the $J$ functions since they are well-behaved at the origin, and if we label the roots with $x_{mn}$ (so that $J_m(x_{mn}) = 0$), then we have the normalized eigenfunctions (for the normalization constant, see Eq. (3.95) in Jackson for instance)

$$R_{mn}(\rho) = \frac{\sqrt{2}}{aJ_{m+1}(x_{mn})} J_m\left(x_{mn}\frac{\rho}{a}\right), \qquad n = 0, 1, 2, \ldots. \tag{11}$$

Thus, our eigenfunctions are just

$$\psi_{kmn}(\mathbf{x}) = \sqrt{\frac{2}{\pi L}} \frac{e^{im\phi}\sin(k\pi z/L)J_m(x_{mn}\rho/a)}{aJ_{m+1}(x_{mn})},$$

and we see that $K^2 = k^2\pi^2/L^2$, $\nu^2 = m^2$, and $K'^2 = x_{mn}^2/a^2$ (in Eq. (8), Eq. (9) and Eq. (10)). From these, we immmediately get the eigenvalues

$$k_{kmn}^2 = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{x_{mn}}{a}\right)^2.$$

When all of this is put together, we get the Green's function

$$G(\mathbf{x}, \mathbf{x}') = \frac{8q}{L} \sum_{m\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')}\sin(k\pi z/L)\sin(k\pi z'/L)J_m(x_{mn}\rho/a)J_m(x_{mn}\rho'/a)}{[(x_{mn}/a)^2 + (k\pi/L)^2]J_{m+1}^2(x_{mn})}.$$

This is in agreement with the Jackson problem 3.23. In principle, the sum over $k$ can be done to get the first solution in Jackson, and the sum over $n$ can be done to get the second solution in Jackson. The first two solutions can also be found by considering the equation for the Green's function and solving directly (solving for $z > z'$ and $z' < z$ and making the solutions continuous at $z = z'$ for the first and for the second solving for $\rho < a$ and $\rho > a$ and making the solutions continuous at $\rho = a$).