ELECTRODYNAMICS

PROBLEM SET 6

due April 1^{st} (really !), before class

I. SPIN OF A ELECTROMAGNETIC WAVE

The angular momentum of the electromagnetic field is defined as

$$\vec{L} = \frac{1}{4\pi c}\int d^3r \; \vec{r}\times(\vec{E}\times\vec{B}) \tag{1}$$

and is a conserved quantity in the absence of matter (you proved this in a previous homework).

a) Consider a field distribution localized in space and show that *vecL* can be split as

$$\vec{L} = \frac{1}{4\pi c}\int d^3r \; \left[\vec{E}\times\vec{A} + E_i(\vec{r}\times\nabla)A_i\right]. \tag{2}$$

The first term is called the spin (since it doesn't depend on the origin of the coordinate system) and the second one the orbital angular momentum.

b) Compute the spin for a linearly, right handed and left handed circularly polarized plane wave. (A plane wave is not a localized field distribution but a similar result would follow from considering a localized wave packet.)

II. FARADAY ROTATION

In class we considered a simple model for dielectrics where the motion of the electron were approximated by charged damped harmonic oscillators. In this problem we extend that model by adding a small external constant magnetic field B_0.

a) Write down the (non-relativistic) equation of motion of a harmonic oscillator under the influence of the electric field of a plane wave and the external, constant B_0 field. Argue that the influence of the magnetic field of the plane wave is small as long as the electron is non-relativistic. Contrary to what was done in class, neglect any damping.

b) Assuming that the B_0 field is in the direction of motion of a circularly polarized plane wave, solve the equations of motion for the electron. Hint: Fourier transform, just like in class.

c) Calculate the polarization density and the dielectric constant as a function of the frequency ω. Notice that they depend on the polarization of the wave. This phenomenon is called *birefringence* .

d) Show that the polarization direction of a linearly polarized wave will change as the wave propagates along the direction of B_0.

Hint: You should find that, for dilute plasmas ($n \approx 1, \omega_0 \approx 0$) , the rate of rotation is proportional to the density of free electrons and it is faster for low frequency waves. These facts, combined, allow astronomers to measure the density and magnetic fields in far away regions of space.

III. BE CREATIVE

You know the drill, create and solve a problem.