Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 1. Energy, forces, and torques on multipoles

Consider a static charge distribution characterized by a charge q, and electric dipole $\mathbf{p}$ and an electric quadrupole $\mathbf{Q}$ in and *external* electric field. Assuming that the external field varies slowly within the size of the charge distribution:

a. Show that the energy of the charge distribution due to the interaction with the external field is given by

$$U = q\phi(0) + \#\mathbf{p}\cdot\mathbf{E} + \#Q_{ij}\partial_i E_j + \cdots, \tag{1}$$

where # are numerical coefficients. Determine the #'s.

b. Using the result above, calculate the energy of one electric dipole in the electric field of another dipole (assuming their separation is much larger than their sizes).

c. What is the analogue expression for the force acting on the charge distribution?

Consider now a localized steady current distribution immersed in an *external* magnetic field varying litle within the size fo the current distribution.

d. What is the expression for the energy of the magnetic dipole?

e. What is the force exerted by the external field on the dipole?

f. What is the torque exterted on the dipole?

Solution

Part a.

Recall that the energy density for the electromagnetic field is

$$u = \tfrac{1}{8\pi}(\mathbf{E}_{\text{ext}} + \mathbf{E})^2,$$

where $\mathbf{E}$ is the field due to the static charge distribution and $\mathbf{E}_{\text{ext}}$ is the external field. Upon expanding this, the only energy term dependent on the interaction between the fields is just

$$u_{\text{int}} = \tfrac{1}{4\pi}\mathbf{E}_{\text{ext}}\cdot\mathbf{E}.$$

If we integrate the interaction energy density, we get the energy (using $\mathbf{E}_{\text{ext}} = \phi_{\text{ext}}$ and $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho$ and integration by parts)

$$\begin{aligned} U &= \frac{1}{4\pi}\int \mathrm{d}^3x\, \mathbf{E}_{\text{ext}}\cdot\mathbf{E} \\ &= \frac{1}{4\pi}\int \mathrm{d}^3x\, \phi_{\text{ext}}\boldsymbol{\nabla}\cdot\mathbf{E} \\ &= \int \mathrm{d}^3x\, \phi_{\text{ext}}(\mathbf{x})\rho(\mathbf{x}). \end{aligned}$$

At this point, we drop the subscript "ext" from the external fields, and we note that

$$\begin{aligned}\phi(\mathbf{x}) &= \phi(0) + x_i\,\partial_i\phi\big|_0 + \tfrac{1}{2}x_i x_j\,\partial_i\partial_j\phi\big|_0 + \cdots \\ &= \phi(0) - \mathbf{x}\cdot\mathbf{E} - \tfrac{1}{2}x_i x_j\partial_i E_j + \cdots .\end{aligned}$$

Note that $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ at the origin (where the external field is being evaluated in the above expression), so $x^2\delta_{ij}\partial_i E_j = 0$, and so we can add to the expression

$$\phi(\mathbf{x}) = \phi(0) - \mathbf{x}\cdot\mathbf{E} - \tfrac{1}{6}(3x_i x_j - x^2\delta_{ij})\partial_i E_j + \cdots .$$

We can put the expansion of $\phi(\mathbf{x})$ into the intergal for U, and using the definitions for the different moments of charge distributions we have

$$U = q\phi(0) - \mathbf{p}\cdot\mathbf{E} - \tfrac{1}{6}Q_{ij}\partial_i E_j + \cdots .$$

Part b.

The electric field due to a dipole as found in homework 3 is

$$\mathbf{E} = -\frac{\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{r^3}.$$

Let us call the distance between dipoles R and the one at the origin is $\mathbf{p}$ while the "external" one is $\mathbf{p}'$. Then, we simply have

$$U = \frac{\mathbf{p}\cdot\mathbf{p}' - 3(\mathbf{p}\cdot\hat{\mathbf{R}})(\mathbf{p}'\cdot\hat{\mathbf{R}})}{R^3}.$$

Part c.

To determine the force, we need to take the gradient of this quantity (with respect to the vector $\mathbf{R}$). This can be accomplished in coordinates as follows

$$U = \frac{\mathbf{p}\cdot\mathbf{p}'}{r^3} - \frac{3p_k x_k p'_j x_j}{r^5},$$

and from this we have the derivative

$$\begin{aligned}\partial_i U &= -\frac{\mathbf{p}\cdot\mathbf{p}'\,x_i}{r^5} - \frac{3p_i(\mathbf{p}'\cdot\mathbf{r}) + 3p'_i(\mathbf{p}\cdot\mathbf{r})}{r^5} + 15\frac{(\mathbf{p}\cdot\mathbf{r})(\mathbf{p}'\cdot\mathbf{r})\,x_i}{r^7} \\ -\boldsymbol{\nabla}U &= \frac{[\mathbf{p}\cdot\mathbf{p}' + 15(\mathbf{p}\cdot\hat{\mathbf{r}})(\mathbf{p}'\cdot\hat{\mathbf{r}})]\hat{\mathbf{r}} + 3(\mathbf{p}'\cdot\hat{\mathbf{r}})\mathbf{p} + 3(\mathbf{p}\cdot\hat{\mathbf{r}})\mathbf{p}'}{r^4}.\end{aligned}$$

With the fact that $\mathbf{F} = -\boldsymbol{\nabla}U = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E})$, we get that the above expression for $-\boldsymbol{\nabla}U$ is the force.

Part d.

From section 5.7 of Jackson, the energy of a magnetic dipole is

$$U = -\mathbf{m}\cdot\mathbf{B}$$

where $\mathbf{m}$ is the magnetic dipole and $\mathbf{B}$ is the field it is immersed in. The magnetic dipole moment can be found from the current as

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')\,\mathrm{d}^3x'.$$

Part e.

The force can be found as just the negative gradient of the energy

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}).$$

Again, see Sec. 5.7 of Jackson for a more complete discussion.

Part f.

The torque is given in the first approximation as

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}(0).$$

Problem 2. Jackson, 4.10

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry the charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ), as shown in the figure.

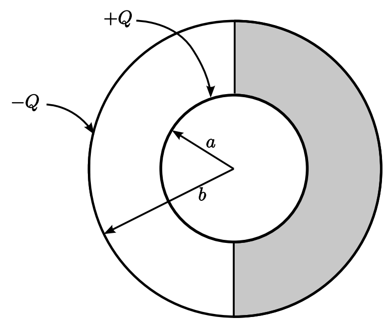

Figure 1: Problem 4.10

(a) Find the electric field everywhere between the spheres.

(b) Calculate the surface-charge distribution on the inner sphere.

(c) Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

Solution

Part (a)

For this part we make the *ansatz* that $\mathbf{E} = E(r)\hat{\mathbf{r}}$ since the potential at $r = a$ and $r = b$ is constant. Of course, we will check to see if this is a valid solution.

Now, we can use the Gauss's theorem for the displacement field $\mathbf{D}$ which tells us

$$\oint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = 4\pi Q_{\text{free}} \tag{2}$$

where Q_{free} is the free charge inclosed by S. We now take S to be a sphere around zero at a radius r in between a and b. We then obtain

$$\begin{aligned} \oint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, \mathrm{d}A &= 4\pi Q \\ \int_{\text{left}} \mathbf{E} \cdot \hat{\mathbf{r}}\, \mathrm{d}A + \int_{\text{right}} \epsilon\mathbf{E} \cdot \hat{\mathbf{r}}\, \mathrm{d}A &= 4\pi Q \\ 2\pi r^2 (1+\epsilon) E(r) &= 4\pi Q. \qquad \text{By our } \textit{ansatz}. \end{aligned}$$

This then gives

$$E(r) = \frac{2Q}{(1+\epsilon) r^2}. \tag{3}$$

Does this actually work though? Clearly in the left and right regions separately $\nabla \cdot \mathbf{D} = 0$ since the same is true for $\mathbf{E}$ and ϵ is constant in both regions. Clearly this is derivable from a potential which makes the potential constant on both the inner sphere and the outer sphere, so that boundary condition is satisfied.

Lastly, we need to check the boundary between the vacuum and the dielectric material. Recall that in the abscence of free charge that the normal component of $\mathbf{D}$ must be continuous. Well, the normal component of $\mathbf{D}$ is zero (between the two mediums), so it must be continuous in this case. Lastly, we need the tangential $\mathbf{E}$ field to be continuous, and this can be clearly seen by the fact that for a given r, $\mathbf{E}$ is constant. Thus, we have satisified all boundary conditions and have a unique solution to the problem!

Part (b)

With the $\mathbf{E}$ field we can easily get the surface charge distribution by noting

$$\mathbf{E} \cdot \hat{\mathbf{n}}\big|_{r=a} = 4\pi\sigma,$$

where σ is on the inner sphere. Thus, we have that

$$\sigma = \frac{Q}{2\pi a^2(1+\epsilon)}.$$

This is the surface charge distribution on the inner sphere.

Part (c)

Now, we can use the $\mathbf{D}$ field to obtain the free charge density. In the region without a dielectric this is the same condition as in (b), so we can focus on the region with dielectric. Here we have that

$$\mathbf{D} \cdot \hat{\mathbf{n}}\big|_{r=a} = 4\pi\sigma_{\text{free}}.$$

So in the region of dielectric $\mathbf{D} = \epsilon\mathbf{E}$ so we obtain

$$\sigma_{\text{free}} = \frac{Q}{2\pi a^2(1+\epsilon)}.$$

Now we know that $\sigma_{\text{free}} + \sigma_{\text{induced}} = \sigma$, so we can solve for σ_{induced} to get

$$\sigma_{\text{induced}} = -\frac{(\epsilon-1)Q}{2\pi a^2(\epsilon+1)}.$$

As expected, it induces a negative charge on the surface.