ELECTRODYNAMICS

PROBLEM SET 2

due February 18, before class

Problem 1.: Angular momentum

One point that was not emphasized in class is that conservation laws are a consequence of symmetries of Nature. For instance, translation symmetry leads to the conservation of momentum and energy, gauge invariance leads to conservation of charge, ... Rotation invariance, being a symmetry of the electromagnetic action, should lead to the existence of a conserved quantity: angular momentum.

a) Show that the angular momentum tensor $M^{\mu\nu\lambda} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$ satisfy $\partial_\lambda M^{\mu\nu\lambda} = 1/c(x^\mu J_\alpha F^{\alpha\nu} - x^\nu J_\alpha F^{\alpha\mu})$.

b) Write down the equation derived above in 3-dimensional notation and describe the physical interpretation of each term.

Problem 2.: Stress-tensor in action

Consider two parallel, infinite charged planes with (surface) charge density equal to σ and $-\sigma$.

a) Calculate the electric field generated by them.

b) Calculate the potential generated by them.

c) Using the result in b), calculate the force (per unit area) between the planes.

d) Calculate the stress tensor.

e) Using the conservation of momentum law, calculate the force between the planes by integrating the stress tensor over a surface separating the two planes.

Problem 3.: Multipole expansion

Consider a linear distribution of charge along the z-axis where the charge density from $z = 0$ to $z = L/2$ is λ and from $z = 0$ to $z = -L/2$ is $-\lambda$.

a) Calculate the potential by direct integration.

b) Calculate the total charge, dipole moment and quadrupole moment of the charge distribution.

c) Calculate the potential using the multipole expansion up to third (quadrupole) order.

d) Compare the results in a) and c) and verify that they agree to the order they should agree.

Problem 4.: Most of vector analysis The components notation ("indices galore") and tensors are also useful in 3 dimensions. The trick is to write the vector product with the help of the 3-D ϵ^{ijk} tensor as

$$(\vec{A} \times \vec{B})^i = \epsilon^{ijk} A^j B^k. \tag{1}$$

a) Show that $\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$ (hint: what else could it be?)

b) Show that $\epsilon^{ijk}\epsilon^{ikm} = 2\delta^{km}$.

c) Show that

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B}) \tag{2}$$

$$\nabla(\vec{A}.\vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + B \times (\nabla \times \vec{A}) + (\vec{A}.)\nabla)\vec{B} + (\vec{B}.)\nabla)\vec{A} \tag{3}$$

$$\nabla.(\vec{A} \times \vec{B}) = \vec{B}.\nabla \times \vec{A} - \vec{A}.\nabla \times \vec{B} \tag{4}$$

$$\nabla.\nabla \times \vec{A} = 0 \tag{5}$$

$$\nabla \times \nabla f = 0 \tag{6}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2 \vec{A}, \tag{7}$$

where $\vec{A}, \vec{B}$ and $\vec{C}$ are vector fields and f a scalar field (all well behaved enough).

Problem 4.: Be creative

a) Invent a problem on the stuff we are studying in class, or related to it. Prizes will be given for fanciness and originality.

b) Solve the problem in a).