Solutions to Final Exam

Physics 604

Dec. 15, 2003

This CLOSED BOOK exam is to be completed in 2 hours. Choose THREE of the four problems — if you attempt all four, be sure to indicate clearly which three should be graded. You must explain your reasoning to receive full credit.

Initialization

```
ClearAll["Global`*"];
Off[General::spell, General::spell1]

$DefaultFont = {"Times", 12};
$TextStyle = {FontFamily → "Times", FontSize → 12, FontSlant → "Italic"};

Needs["Graphics`Master`"];
Needs["Utilities`Notation`"];
```

▼ Problem 1 (points per part: 4, 4, 2)

Sound waves in a pipe of slowly varying cross section $A[x]$ satisfy the equation

$$\frac{A[x]}{c^2}\frac{\partial^2\psi[x,t]}{\partial t^2}-\frac{\partial}{\partial x}\left(A[x]\frac{\partial\psi[x,t]}{\partial x}\right)==0$$

Consider an *exponential horn* with $A[x]\propto e^{a x}$.

a) Show that there exist travelling waves of the form

$$\psi[x,t]=e^{-b x}\,\text{Cos}[k x-\omega t]$$

where b is independent of ω and determine the dispersion relation $k[\omega]$.

b) Describe this solution. What happens for low frequencies? Develop and describe an alternative solution, if needed, for low frequencies.

c) Evaluate and sketch the phase and group velocities for propagating waves as functions of either k or ω (your choice).

a)

We simply substitute the proposed solution into the wave equation to obtain

$$-\frac{\omega^2}{c^2}\,\text{Exp}[(a-b)x]\,\text{Cos}[k x-\omega t]+\frac{\partial}{\partial x}\left(\text{Exp}[(a-b)x]\,(b\,\text{Cos}[k x-\omega t]+k\,\text{Sin}[k x-\omega t])\right)==0$$

or

$$-\frac{\omega^2}{c^2}\operatorname{Cos}[k\,x-\omega\,t]+(a-b)\,(b\operatorname{Cos}[k\,x-\omega\,t]+k\operatorname{Sin}[k\,x-\omega\,t])+(-k\,b\operatorname{Sin}[k\,x-\omega\,t]+k^2\operatorname{Cos}[k\,x-\omega\,t])==0$$

Collecting the coefficients of the sine and cosine functions provides the two equations

$$-\frac{\omega^2}{c^2}+(a-b)\,b+k^2==0$$
$$(a-b)\,k-k\,b==0$$

such that

$$b=\frac{a}{2}$$
$$-\frac{\omega^2}{c^2}+\frac{a^2}{4}+k^2==0\implies k=\sqrt{\frac{\omega^2}{c^2}-\frac{a^2}{4}}$$

This solution is an attenuated travelling wave where the attenuation length, b^{-1}, is independent of frequency. Hence, the balance between high and moderately low frequencies is preserved as the sound propagates.

b)

However, there is a low-frequency cut-off, $\omega_c=a\,c/2=b\,c$, below which waves do not propagate because k becomes complex. For very low frequencies, we substitute a trial solution of the form

$$\psi[x,t]=e^{-b\,x}\operatorname{Cos}[\omega\,t]\implies-\frac{\omega^2}{c^2}+b(a-b)==0$$

to obtain

$$b=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{\omega^2}{c^2}}$$

for standing waves. Both damping coefficients are positive and must be retained, but after some distance the smaller one will dominate and determines the maximum penetration of low-frequencies vibrations into the horn. Notice that at $\omega\to 0$ there is a static solution with $b\to 0$ that is not attenuated.

c)

The phase and group velocities are given by

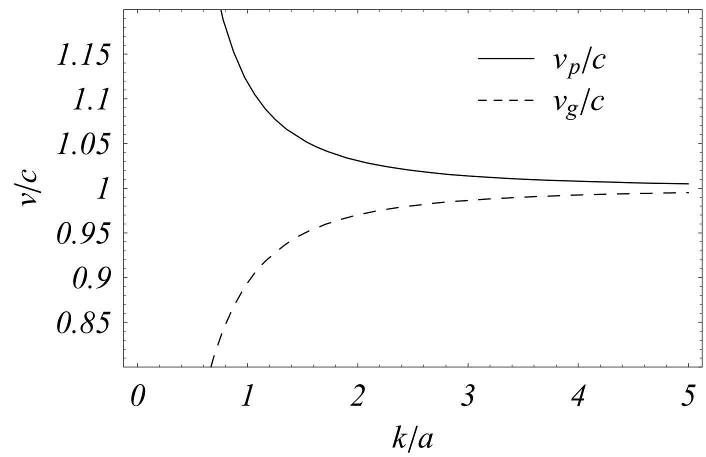
$$v_p=\frac{\omega}{k}=c\sqrt{1+\frac{a^2}{4\,k^2}}$$
$$\omega\,d\omega==c^2\,k\,dk\implies v_g=\frac{d\omega}{dk}=\frac{c^2}{v_p}$$

such that

$$v_p\,v_g=c^2$$

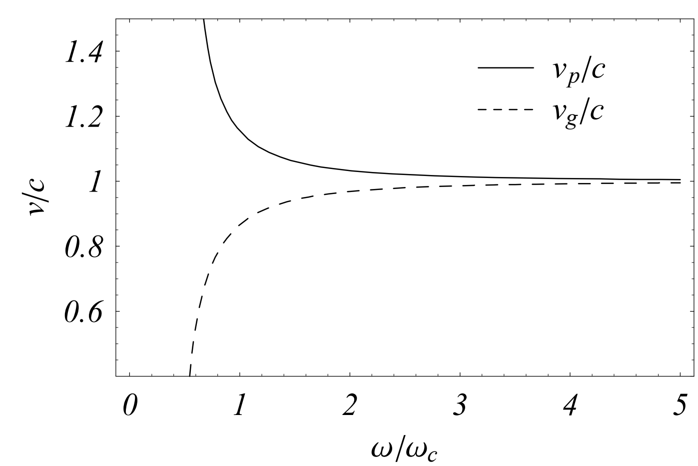
These velocities are sketched below as functions of k/a. Notice that v_p diverges and v_g vanishes at $k=0$.

```
Plot[Evaluate[{(1 + 1/(4 k^2))^(1/2), (1 + 1/(4 k^2))^(-1/2)}],
  {k, 0.6, 5}, PlotRange → {Automatic, {0.8, 1.2}}, Frame → True,
  FrameLabel → {"k/a", "v/c"}, PlotStyle → {{}, Dashing[{0.02, 0.02}]},
  PlotLegend → {"vp/c", "vg/c"}, LegendPosition → {0.3, 0.3},
  LegendSize → {0.6, 0.25}, LegendShadow → None];
```



It may also be of interest to sketch these velocities as functions of ω/ω_c instead. Notice that v_p diverges and v_g vanishes at ω_c, and are complex for smaller frequencies.

```
Plot[Evaluate[{ω (ω^2 - 1/4)^(-1/2), (ω^2 - 1/4)^(1/2) / ω}],
  {ω, 0.5, 5}, PlotRange → {Automatic, {0.4, 1.5}}, Frame → True,
  FrameLabel → {"ω/ωc", "v/c"}, PlotStyle → {{}, Dashing[{0.02, 0.02}]},
  PlotLegend → {"vp/c", "vg/c"}, LegendPosition → {0.3, 0.3},
  LegendSize → {0.6, 0.25}, LegendShadow → None];
```



▼ Problem 2 (10 points)

Use the Laplace transform to solve the initial-value problem

$$y'[t] + \int_0^t y[\tau]\, d\tau == e^{-t}, \; y[0] == 1$$

solution

The Laplace transform of this integro-differential equation reads

$$s\,\tilde{y} - 1 + s^{-1}\,\tilde{y} == (s+1)^{-1} \implies \tilde{y} = \frac{s\,(s+2)}{(s+1)\,(s^2+1)}$$

The solution transform is resolved into partial fractions according to

$$\frac{A}{s+1} + \frac{B}{s+i} + \frac{C}{s-i} == \frac{s\,(s+2)}{(s+1)\,(s^2+1)} \implies A\,(s^2+1) + B\,(s+1)\,(s-i) + C\,(s+1)\,(s+i) == s^2 + 2\,s$$

which yields the system of equations

$$A + B + C == 1$$
$$B\,(1-i) + C\,(1+i) == 2$$
$$A - i\,B + i\,C == 0$$

The second gives

$$B = \frac{2-(1+i)\,C}{1-i}$$

and substitution into the third gives

$$(1-i)\,A - 2\,i + i\,(1+i)\,C + i\,(1-i)\,C == 0 \implies A = \frac{2\,i}{1-i}\,(1-C)$$

Finally, substitution into the first equation gives

$$2\,i\,(1-C) + 2 - (1-i)\,C + (1-i)\,C == 0 \implies C = \frac{1+3\,i}{4\,i} = \frac{3-i}{4}$$

such that

$$A = \frac{2\,i}{1-i}\left(\frac{1+i}{4}\right) = \frac{i}{4}\,(1+i)^2 = -\frac{1}{2}$$
$$B = \frac{8-(1+i)\,(3-i)}{4\,(1-i)} = \frac{8-(4+2\,i)}{4\,(1-i)} = \frac{(4-2\,i)\,(1+i)}{8} = \frac{3+i}{4}$$

Thus,

$$\tilde{y} = \frac{s\,(s+2)}{(s+1)\,(s^2+1)} = -\frac{1}{2}\,\frac{1}{s+1} + \frac{3+i}{4}\,\frac{1}{s+i} + \frac{3-i}{4}\,\frac{1}{s-i}$$

can be inverted using

$$\frac{1}{s+\alpha} \longrightarrow e^{-\alpha\,t} \implies y[t] = -\frac{1}{2}\,e^{-t} + \frac{3+i}{4}\,e^{-i\,t} + \frac{3-i}{4}\,e^{i\,t}$$

Therefore, we obtain

$$y[t] = \frac{1}{2}(3\,\text{Cos}[t] + \text{Sin}[t] - e^{-t})$$

▼ Problem 3 (10 points)

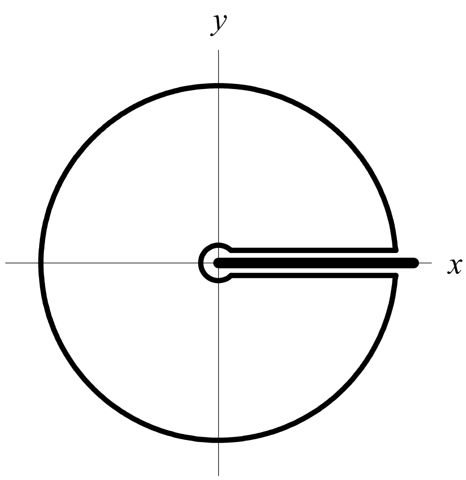
Consider an integral of the form

$$\int_0^\infty g[x]\,dx$$

where $g[x]$ is well-behaved on the positive real axis but is not symmetric with respect to the sign of x, such that the integration interval cannot be extended for use with a great semicircle. If the corresponding $g[z]$ is a meromorphic function that decreases sufficiently rapidly for $z \to \infty$ and is sufficiently small at the origin, one can often use $f[z] = g[z]\,\text{Log}[z]$ with a branch cut on the positive real axis and a *PacMan* contour of the form shown below. Apply this method for the following integral.

$$\int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)}$$

For this integral one could resolve the integrand into partial fractions instead, but to obtain full credit you must employ the method described above, which can be useful for more complicated problems that are not suitable for partial-fraction decomposition. Be sure to justify neglect of any portions of the contour that do not contribute.



solution

Let

$$f[z] = \frac{\text{Log}[z]}{(z+1)(z^2+2z+2)} = \frac{\text{Log}[z]}{(z+1)(z+1+i)(z+1-i)}$$

with a branch cut on the positive real axis, such that

$$z = x + i\,\varepsilon \implies f[z] = \frac{\text{Log}[x]}{(x+1)(x^2+2x+2)}$$
$$z = x - i\,\varepsilon \implies f[z] = \frac{\text{Log}[x] + 2\pi i}{(x+1)(x^2+2x+2)}$$

For large R we use

$$z = R\,e^{i\theta} \implies \left| f[z] \frac{dz}{d\theta} \right| \simeq \frac{|\text{Log}[R] + i\,\theta|}{R^2} \to 0$$

while near the origin

$$z = \varepsilon\,e^{i\theta} \implies \left| f[z] \frac{dz}{d\theta} \right| \approx |\varepsilon\,(\text{Log}[\varepsilon] + i\,\theta)| \to 0$$

Thus, we can neglect both circular portions and obtain

$$\int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)} = -\frac{1}{2\pi i} \oint_C \frac{\text{Log}[z]}{(z+1)(z+1+i)(z+1-i)}\,dz$$

The right-hand side can now be evaluated as the sum of the residues for $z = -1,\ -1-i,\ -1+i$, such that

$$\int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)} = -\left(\frac{\text{Log}[-1]}{(i)(-i)} + \frac{\text{Log}[-1-i]}{(-i)(-2i)} + \frac{\text{Log}[-1+i]}{(i)(2i)}\right)$$
$$= -\left(i\,\pi - \frac{1}{2}\left(\text{Log}\left[\sqrt{2}\right] + \frac{5\pi}{4}\,i\right) - \frac{1}{2}\left(\text{Log}\left[\sqrt{2}\right] + \frac{3\pi}{4}\,i\right)\right)$$

Therefore,

$$\int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)} = \text{Log}\left[\sqrt{2}\right]$$

▼ Problem 4 (5 points per part)

The curved surface of a cylinder of radius a is grounded while the endcaps at $z = \pm L/2$ are maintained at opposite potentials $\psi[\xi, \phi, \pm L/2] = \pm V[\xi, \phi]$.

a) Develop an expansion for the electrostatic potential $\psi[\xi, \phi, z]$ within the cylinder and express the coefficients in terms of the appropriate integral over $V[\xi, \phi]$.

b) Determine the coefficients for the simple case $V[\xi, \phi] = V_0$ where V_0 is constant.

Possibly useful information:

$$\nabla^2 = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{d^2 R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(k^2 - \frac{m^2}{\xi^2}\right) R == 0 \implies R = A\,J_m[k\,\xi] + B\,N_m[k\,\xi]$$

$$J_m[k_{m,n}\,a] = 0 \implies \int_0^a J_m[k_{m,n}\,\xi]^2\,\xi\,d\xi = \frac{a^2}{2} J'_m[k_{m,n}\,a]^2$$

$$m \geq 1 \implies \frac{d}{dx}\left(x^m J_m[x]\right) = x^m J_{m-1}[x]$$

a)

The Laplacian takes the form

$$\nabla^2 = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

in cylindrical coordinates. Laplace's equation

$$\nabla^2\psi == 0 \qquad \psi[\xi, \phi, z] = R[\xi]\,\Phi[\phi]\,Z[z]$$

with boundary conditions

$$R[0] \text{ finite}, \quad R[a] == 0$$
$$\Phi[\phi + 2\pi] == \Phi[\phi]$$
$$Z\left[-\frac{L}{2}\right] == -Z\left[\frac{L}{2}\right]$$

then separates into

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} == -m^2 \implies \Phi = e^{i m \phi}, \quad m = 0, \pm 1, \pm 2, \cdots$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} == k^2 \implies Z = \text{Sinh}[k z]$$

$$\frac{d^2 R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(k^2 - \frac{m^2}{\xi^2}\right)R == 0 \implies R = J_m[k\,\xi]$$

where J_m is the Bessel function of the first kind, the one that is finite at the origin. Note that although the Bessel equation is insensitive to the sign of m, the conventional definition $J_{-m}[\xi] = (-)^m J_m[\xi]$ simplifies some of the relationships among Bessel functions. The boundary condition at the curved surface requires $k \to k_{m,n}$ where the index n enumerates the roots of

$$J_m[k_{m,n}\,a] == 0$$

Thus, the general solution takes the form

$$\psi[\xi, \phi, z] = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\psi_{m,n}\,J_m[k_{m,n}\,\xi]\,\text{Sinh}[k_{m,n}\,z]\,e^{i m \phi}$$

where the $\psi_{n,m}$ are numerical coefficients obtained by matching the boundary conditions at $z = \pm L/2$ according to

$$\psi_{m,n} = \left(\pi\,a^2\,J_m'[k_{m,n}\,a]^2\,\text{Sinh}[k_{m,n}\,L/2]\right)^{-1}\int_0^{2\pi}d\phi\int_0^a d\xi\,\xi\,e^{-i m \phi}\,J_m[k_{m,n}\,\xi]\,V[\xi, \phi]$$

b)

The expansion coefficients for $V[\xi, \phi] = V_0$ require the integrals

$$\int_0^{2\pi}d\phi\;e^{-i m \phi} = 2\pi\,\delta_{m,0}$$

$$\int_0^a J_0[k_{0,n}\,\xi]\,\xi\,d\xi = \frac{1}{k_{0,n}{}^2}\int_0^{k_{0,n}\,a}J_0[x]\,x\,dx = \frac{1}{k_{0,n}{}^2}\int_0^{k_{0,n}\,a}\frac{d}{dx}(x\,J_1[x])\,dx = \frac{a}{k_{0,n}}J_1[k_{0,n}\,a]$$

Thus, the electrostatic potential becomes

$$\psi[\xi, \phi, z] = \frac{2\,V_0}{a} \sum_{n=1}^{\infty} \frac{J_0[k_{0,n}\,\xi]}{k_{m,n}\,J_1[k_{0,n}\,a]} \frac{\mathrm{Sinh}[k_{0,n}\,z]}{\mathrm{Sinh}[k_{0,n}\,L/2]}$$