7.11. Under the conditions of this problem, the summation in eqn. (7.1.2) has to be carried out over the states of the internal spectrum as well as over the translational states. Expression (7.1.16) is then replaced by

$$N_e = (N_e)_0 + (N_e)_1 = \frac{V}{\lambda^3} g_{3/2} \left\{ \exp \left( \frac{\mu}{kT} \right) \right\} + \frac{V}{\lambda^3} g_{3/2} \left\{ \exp \left( \frac{\mu - \varepsilon_1}{kT} \right) \right\}.$$

The critical temperature $T_c$ is then determined by the condition

$$\frac{V}{\lambda^3_e} g_{3/2}(1) + \frac{V}{\lambda^3_e} g_{3/2}(x) = N, \quad \text{where} \quad x = e^{-\varepsilon_1/kT}.$$

(1)

For $x \ll 1$, $g_{3/2}(x) \approx x$ and eqn. (1) gives

$$\lambda^3_e \approx \frac{V}{N} \left[ \zeta(3/2) + x \right].$$

Comparing this with the standard result $(\lambda^0_e)^3 = \left( \frac{V}{N} \right) \zeta(3/2)$, we get

$$\frac{T_c}{T_c^0} \approx \left( \frac{\lambda^0_e}{\lambda_e} \right)^2 = \left[ 1 + \frac{x}{\zeta(3/2)} \right]^{-2/3} \approx 1 - \frac{2/3}{\zeta(3/2)} x \approx 1 - \frac{2/3}{\zeta(3/2)} e^{-\varepsilon_1/kT^0}.$$

For $x \leq 1$, on the other hand, $g_{3/2}(x) \approx \zeta(3/2) - 2\pi^{1/2} (-\ln x)^{1/2}$; eqn. (1) now gives

$$\lambda^3_e \approx \frac{2V}{N} \left[ \zeta(3/2) - \pi^{1/2} \left( \frac{\varepsilon_1}{kT_c^0} \right)^{1/2} \right], \quad \text{whence}$$

$$\frac{T_c}{T_c^0} \approx 2 \left[ 1 - \frac{\pi^{1/2}}{\zeta(3/2)} \left( \frac{\varepsilon_1}{kT_c^0} \right)^{1/2} \right]^{-2/3} \approx 2^{-2/3} \left[ 1 + \frac{2}{3} \frac{\pi^{1/2}}{\zeta(3/2)} \left( \frac{\varepsilon_1}{kT_c^0} \right)^{1/2} \left( \frac{\varepsilon_1}{kT_c^0} \right)^{1/2} \right].$$
7.13. It is straightforward to see that for a Bose gas in two dimensions

\[ N_e = \int_0^1 \frac{1}{z^2 e^{\lambda e} - 1} \frac{A \cdot 2 \pi m dp}{h^2} = \frac{A \cdot 2 \pi m k T}{h^2} \int_0^1 \frac{dx}{z^2 e^{\lambda x} - 1} = \frac{A}{\lambda^2} g_1(z), \]

while

\[ N_0 = \frac{z}{1 - z}. \]

Since Bose-Einstein condensation requires that \( z \to 1 \), the critical temperature \( T_c \), by the usual argument, is given by

\[
\left( \frac{N}{A} \right) \lambda^2_c = g_1(1) = \infty \quad \text{[for } g_1(z) = -\ln(1 - z)\text{]}.
\]

It follows that \( T_c = 0 \).

More accurately, the phenomenon of condensation requires that both \( N_e \) and \( N_0 \) be of order \( N \).

This means that, while \( z \approx 1 \), \( (1 - z) \) be of order \( N^{-1} \) and hence \( \lambda^2 \) be of order \( (A \ln N / N) \). Since the ratio \( (A / N) \approx \ell^2 \), the condition for condensation takes the form \( (\lambda^2 / \ell^2) = O(\ell n N) \). It follows that

\[ T = \frac{\hbar^2}{2 \pi m \lambda^2} - \frac{\hbar^2}{mk \ell^2} \frac{1}{\ell n N}. \]
7.14. With energy spectrum $\varepsilon = Ap^s$, the density of states in the system is given by, see formula (C.7b),

$$a(\varepsilon)d\varepsilon = \frac{V}{h^n} \frac{2\pi^{n/2}}{\Gamma(n/2)} p^{n-1} dp = \frac{V}{h^n} \frac{2\pi^{n/2}}{sA^{n/2}\Gamma(n/2)} \varepsilon^{(n/s)-1} d\varepsilon .$$  \hspace{1cm} (1)

This leads to the expression

$$N - N_0 = \frac{V}{h^n} \frac{2\pi^{n/2}}{sA^{n/2}\Gamma(n/2)} \int_0^{\varepsilon^{(n/s)-1}} z^{-1} e^{\varepsilon - 1} \varepsilon d\varepsilon$$

$$= \frac{V}{h^n} \frac{2\pi^{n/2}}{sA^{n/2}\Gamma(n/2)} (kT)^{(n/s)} g_{nls}(z),$$  \hspace{1cm} (2)

while $N_0 = z/(1 - z)$. Similarly,

$$P = \frac{1}{h^n} \frac{2\pi^{n/2}}{sA^{n/2}\Gamma(n/2)} (kT)^{(n/s)+1} g_{nls+1}(z).$$  \hspace{1cm} (3)

Next, following the derivation of eqn. (7.1.11), we get

$$U = kT^2 \left( \frac{\partial}{\partial T} \left( \frac{PV}{kT} \right) \right)_{\varepsilon,V} = \frac{n}{s} PV,$$

so that $P = sU / nV$.

The onset of Bose-Einstein condensation requires that $z \rightarrow 1$ at a finite temperature $T_c$. A glance at eqn. (2) tells us that this will happen only if $n > s$ and that the critical temperature $T_c$ will then be determined by the equation

$$N = \frac{V}{h^n} \frac{2\pi^{n/2}}{sA^{n/2}\Gamma(n/2)} (kT_c) \xi \left( \frac{n}{s} \right).$$  \hspace{1cm} (5)

For $T < T_c$, $N_s$ will be equal to $N(T / T_c)^{n/s}$ while $N_0$ will be given by the balance $(N - N_s)$.

To study the specific heats we first observe, from eqns. (2) - (4), that for $T > T_c$ (when $N_0 \ll N$)

$$U = \frac{n}{s} NkT \cdot \frac{g_{nls+1}(z)}{g_{nls}(z)}. $$  \hspace{1cm} (6)

Next, using eqns. (2) and (3), and the recurrence relation (D.10), we get

$$\frac{1}{z} \left( \frac{\partial z}{\partial T} \right)_V = -\frac{n}{s} \frac{1}{T} \frac{g_{nls}(z)}{g_{nls-1}(z)} \quad \text{and} \quad \frac{1}{z} \left( \frac{\partial z}{\partial T} \right)_P = -\left( \frac{n}{s} + 1 \right) \frac{g_{nls+1}(z)}{g_{nls}(z)}. $$  \hspace{1cm} (7)

It is now straightforward to show that

$$\frac{C_V}{Nk} = \frac{n}{s} \left( \frac{n}{s} + 1 \right) \frac{g_{nls+1}(z)}{g_{nls}(z)} - \left( \frac{n}{s} \right)^2 \frac{g_{nls}(z)}{g_{nls-1}(z)} $$  \hspace{1cm} (8)

and

$$\frac{C_p}{Nk} = \left( \frac{n}{s} + 1 \right)^2 \left[ \frac{g_{nls+1}(z)}{g_{nls}(z)} \right]^2 \frac{g_{nls-1}(z)}{g_{nls}(z)} - \frac{n}{s} \left( \frac{n}{s} + 1 \right) \frac{g_{nls+1}(z)}{g_{nls}(z)}. $$  \hspace{1cm} (9)

The limiting cases suggested in the problem follow quite easily.
According to Sec. 7.3,

\[ C_v(T) = \int_0^\infty \frac{\hbar \omega}{e^{\hbar \omega / kT} - 1} g(\omega) d\omega, \quad \text{while} \quad C_v(\infty) = \int_0^\infty k g(\omega) d\omega. \]

It follows that

\[ \int_0^\infty \{ C_v(\infty) - C_v(T) \} dT = \int_0^\infty \left[ kT - \frac{\hbar \omega}{e^{\hbar \omega / kT} - 1} \right] g(\omega) d\omega. \]

It is easy to show that

\[ \lim_{T \to \infty} \frac{\hbar \omega}{e^{\hbar \omega / kT} - 1} = kT - \frac{1}{2} \hbar \omega; \]

see page 69 as well as Fig. 3.4 of the text. The integral on the right-hand side then becomes

\[ \int_0^\infty \frac{1}{2} \hbar \omega \cdot g(\omega) d\omega, \]

which is indeed equal to the zero-point energy of the solid.

The physical interpretation of this result lies in noting that the actual amount of heat required to raise the temperature of a solid is less than the value predicted classically because the solid already possesses a finite amount of energy even at \( T = 0K. \)