Spring 2003 Physics 603 O. W. Greenberg

Solutions to final exam

1. (a) $\Omega = \dfrac{N!}{N_+!\,N_-!}$, $N_+ + N_- = N$, $(N_+ - N_-)\epsilon = E$

(b) $\begin{pmatrix} 1 & 1 \\ \epsilon & -\epsilon \end{pmatrix}\begin{pmatrix} N_+ \\ N_- \end{pmatrix} = \begin{pmatrix} N \\ E \end{pmatrix}$, $\Delta = -2\epsilon$

$$N_+ = -\frac{1}{2\epsilon}\begin{vmatrix} N & 1 \\ E & -\epsilon \end{vmatrix} = \frac{N\epsilon + E}{2\epsilon}$$

$$N_- = -\frac{1}{2\epsilon}\begin{vmatrix} 1 & N \\ \epsilon & E \end{vmatrix} = \frac{N\epsilon - N}{2\epsilon}$$

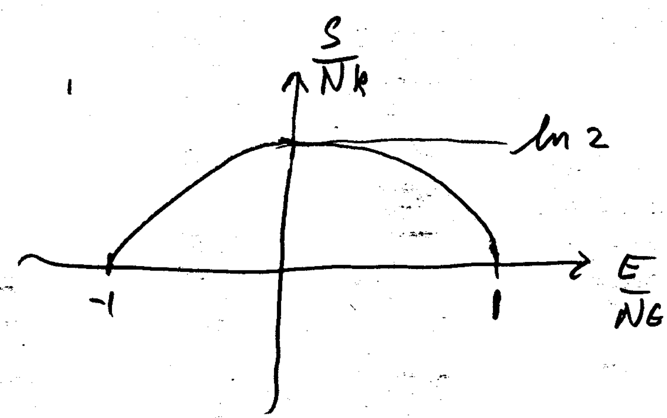
$$S(N,E) = k\ln\Omega = k[\underbrace{N}_{N_+ + N_-}\ln N - N_+\ln N_+ - N_-\ln N_-]$$

$$= k[N_+\ln\frac{N}{N_+} + N_-\ln\frac{N}{N_-}]$$

$$= k[\frac{N\epsilon + E}{2\epsilon}\ln\frac{2N\epsilon}{N\epsilon + E} + \frac{N\epsilon - E}{2\epsilon}\ln\frac{2N\epsilon}{N\epsilon - E}]$$

(c) $$\frac{S}{Nk} = \frac{1+\frac{E}{N\epsilon}}{2}\ln\frac{2}{1+\frac{E}{N\epsilon}} + \frac{1-\frac{E}{N\epsilon}}{2}\ln\frac{2}{1-\frac{E}{N\epsilon}}$$

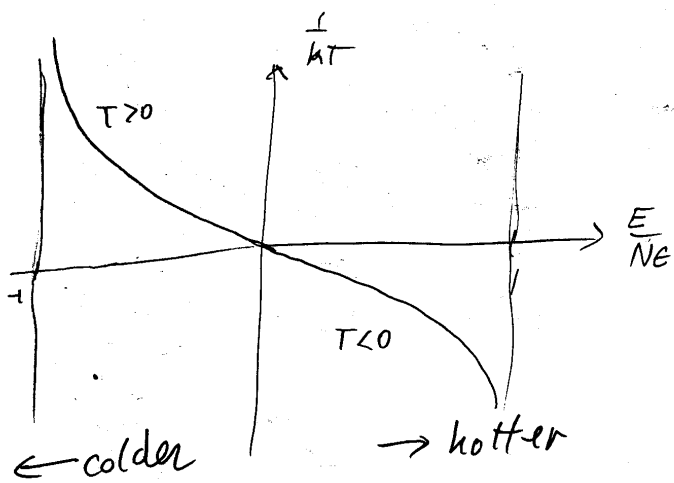
$\frac{S}{Nk}(-1) = \ln 1 = 0$, $\frac{S}{Nk}(0) = \frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 = \ln 2$, $\frac{S}{Nk}(1) = 0$



(d) $$\frac{1}{kT} = \frac{1}{k}\left(\frac{\partial S}{\partial E}\right)_N = \frac{\frac{1}{\epsilon}}{2}\ln\frac{2}{1+\frac{E}{N\epsilon}} + \frac{1+\frac{E}{N\epsilon}}{2}(-1)\frac{\frac{1}{\epsilon}}{1+\frac{E}{N\epsilon}}$$

$$+ \frac{-\frac{1}{\epsilon}}{2}\ln\frac{2}{1-\frac{E}{N\epsilon}} + \frac{1-\frac{E}{N\epsilon}}{2}(-1)\frac{-\frac{1}{\epsilon}}{1-\frac{E}{N\epsilon}}$$

$$= \frac{1}{2\epsilon}\ln\frac{1-\frac{E}{N\epsilon}}{1+\frac{E}{N\epsilon}} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2\epsilon}\ln\frac{1-\frac{E}{N\epsilon}}{1+\frac{E}{N\epsilon}}$$



(e) $E < 0$

(f) $T<0$, greater occupancy in upper state.

(g) $P = -\left(\frac{\partial E}{\partial V}\right)_{N,S} = 0$. When there are no translational modes, $P = 0$.

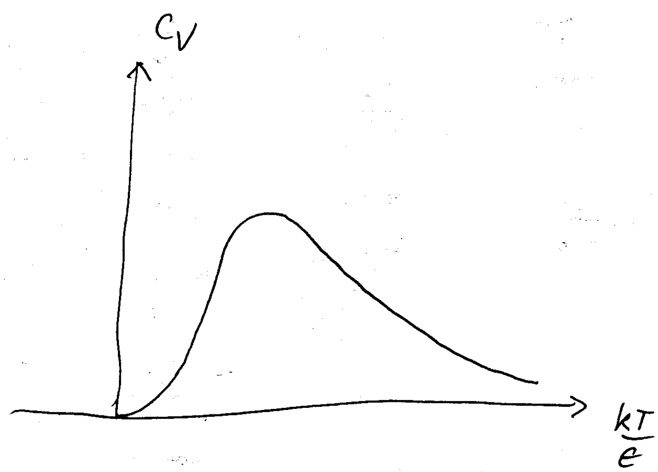
(h) From (d), $kT = \dfrac{2\epsilon}{\ln\frac{1-\frac{E}{N\epsilon}}{1+\frac{E}{N\epsilon}}}$, $E = -N\epsilon \tanh\dfrac{\epsilon}{kT}$

$$C_V = \frac{\partial E}{\partial T} = -N\epsilon\ \mathrm{sech}^2 \frac{\epsilon}{kT}\left(-\frac{\epsilon}{kT^2}\right) = \frac{N\epsilon^2}{kT^2}\,\frac{1}{\cosh^2\frac{\epsilon}{kT}}$$

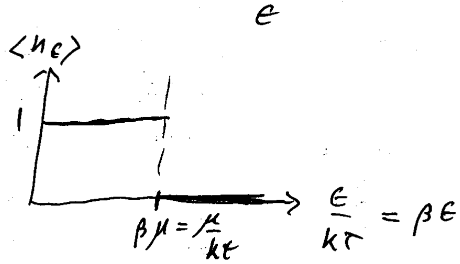
(g) For $T \to 0^+$, $C_V \to \frac{N\epsilon^2}{4kT^2}\, e^{-\frac{2\epsilon}{kT}} \to 0$ rapidly.

For $T \to \infty$, $C_V \to \frac{N\epsilon^2}{kT^2} \to 0$ as $\frac{1}{T^2}$

(h)



2. (a) $\langle n_\epsilon \rangle = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$.



(b)
$$N = (2S+1)\int \frac{d^D p\, d^D q}{h^D}\,\frac{1}{\frac{1}{z}e^{\beta\epsilon}+1} \qquad z = e^{\beta\mu}$$

$$= (2S+1)\frac{L^D}{h^D}\,\Omega_D \int_0^\infty p^{D-1}\,dp\,\frac{1}{\frac{1}{z}\,e^{\beta\frac{p^2}{2m}}+1}$$

Let $x = \frac{\beta p^2}{2m}$

$p = \sqrt{\frac{2mx}{\beta}}$

$$N = (2s+1)\frac{L^D}{h^D}\Omega_D\left(\frac{2m}{\beta}\right)^{\frac{D}{2}}\int_0^\infty \frac{x^{\frac{D}{2}-1}\,dx}{\frac{1}{z}e^x+1}$$

h has dimensions of $dp\,dq$.

(c) If $\beta(\epsilon-\mu)\sim 1$, then $\langle n_\epsilon\rangle \sim \frac{1}{e+1} \sim \frac{1}{3.7}$, so the width of excited particles is $\sim \Delta(\beta\epsilon)=1$, or $\frac{\epsilon}{kT}\sim 1$. Then the fraction of particles excited is $\sim \frac{kT}{\epsilon_F}$. The average excited particle gets energy $\sim kT$, so $\Delta E \sim \frac{N(kT)^2}{\epsilon_F}$.

(d) The low-T dependence is

$$\frac{c_V}{Nk} \sim \frac{2kT}{\epsilon_F},$$ linear in T. The dimension D enters in a more careful calculation.

(e) By equipartition, each quadratic term contributes $\frac{1}{2}kT$ per particle to E. In D dimensions, $\frac{p^2}{2m} = \frac{1}{2m}\sum_1^D p_i^2$, so $\frac{C_V}{Nk} = \frac{D}{2}$. (The $2s+1$ factor cancels in $\frac{E}{N}$ or $\frac{C_V}{N}$.)

3. (a) $H\{\sigma_i\} = -J\sum_{i=1}^N \sigma_i\sigma_{i+1} - \frac{\mu B}{2}\sum_{i=1}^N(\sigma_i+\sigma_{i+1})$, where $\sigma_{N+1}\equiv\sigma_1$.

(b) $Q_N(B,T) = \sum_{\{\sigma_i=\pm\}} \exp\left\{\sum_{i=1}^N J\sigma_i\sigma_{i+1} + \frac{1}{2}\mu B(\sigma_i+\sigma_{i+1})\right\}$

(c) $\langle\sigma_i|\mathbb{P}|\sigma_{i+1}\rangle = \begin{pmatrix} e^{\beta(J+\mu B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)}\end{pmatrix}$

$Q_N = \mathrm{tr}\,\mathbb{P}^N$

(d) $$\begin{vmatrix} e^{\beta(J+\mu B)} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} - \lambda \end{vmatrix} = 0,$$

$$\lambda^2 - 2e^{\beta J}\cosh\beta\mu B \cdot \lambda + 2\sinh 2\beta\mu B = 0$$

$$\lambda_\pm = \frac{1}{2}\left[2e^{\beta J}\cosh\beta\mu B \pm \sqrt{4e^{2\beta J}\cosh^2\beta\mu B - 8\sinh 2\beta\mu B}\right]$$

$$= e^{\beta J}\cosh\beta\mu B \pm \sqrt{e^{2\beta J}\sinh^2\beta\mu B + e^{-2\beta J}}$$

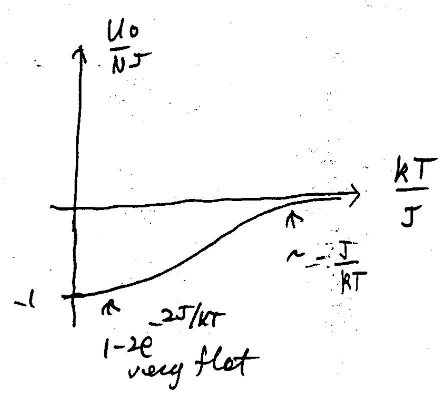
(e) $\frac{\lambda_-}{\lambda_+} < 1$, so for $N \to \infty$, $\mathrm{tr}\, P^N \to \lambda_+^N$.

$$Q_N = \left[e^{\beta J}\cosh\beta\mu B + \sqrt{e^{2\beta J}\sinh^2\beta\mu B + e^{-2\beta J}}\right]^N$$

(f) $$\frac{U_0}{NJ} = -\frac{1}{NJ}\left(\frac{\partial}{\partial\beta}\ln Q_N\right)_{B=0} = -\frac{1}{J}\frac{\partial}{\partial\beta}\ln\left[e^{\beta J} + \sqrt{e^{-2\beta J}}\right]$$

$$= -\frac{1}{N}\frac{\partial}{\partial\beta}\ln(2\cosh\beta J) = -\frac{\sinh\beta J}{\cosh\beta J} = -\tanh\beta J$$

(g)



$T \to 0^+$, $\tanh\frac{J}{kT} = \frac{e^{J/kT} - e^{-J/kT}}{e^{J/kT} + e^{-J/kT}}$

$= \frac{1 - e^{-2J/kT}}{1 + e^{-2J/kT}} \to 1 - 2e^{-2J/kT}$

$T \to \infty$, $\tanh\frac{J}{kT} \to \frac{J}{kT}$

(h) At $T \to 0^+$, all spins are parallel, either all 1 or all -1, so energy is minimum. At $T \to \infty$, spins are random.
ferromagnet

(i) $$\bar{M} = \frac{1}{\beta}\frac{\partial}{\partial B} \ln Q_N = \frac{N}{\beta} \frac{\beta\mu e^{\beta J}\sinh\beta\mu B + \dfrac{2\beta\mu e^{2\beta J}\cosh\beta\mu B \sinh\beta\mu B}{2\sqrt{e^{2\beta J}\sinh^2\beta\mu B + e^{-2\beta J}}}}{e^{\beta J}\cosh\beta\mu B + \sqrt{e^{2\beta J}\sinh^2\beta\mu B + e^{-2\beta J}}}$$

$$= N\mu \frac{\sinh\beta\mu B}{\sqrt{\sinh^2\beta\mu B + e^{-4\beta J}}}$$

$$\lim_{B \to 0} \bar{M} = N\mu \frac{0}{\sqrt{0 + e^{-4\beta J}}} = 0, \quad T > 0,$$

$$= \lim_{T \to 0} N\mu \frac{\sinh\beta\mu B}{\sinh\beta\mu B} = N\mu, \quad T = 0,$$

So get a ferromagnetic phase transition at exactly $T = 0$.