1. Consider $N$ atoms, each of which can be either in its ground state or an excited state of energy $\Delta$. Suppose that $N_1$ atoms are in the excited state (and so $N_0 = N - N_1$ in the ground state), so that the energy $E = N_1 \Delta$.

a) Find the number of configurations giving $N_1$ excited atoms and use the Stirling approximation to show that the entropy is

$$S(E, N) = -N k_B \left[ \left( \frac{E}{N \Delta} \right) \ln \left( \frac{E}{N \Delta} \right) + \left( 1 - \frac{E}{N \Delta} \right) \ln \left( 1 - \frac{E}{N \Delta} \right) \right]$$

b) Is this entropy extensive? Justify your answer.

c) Identifying $E$ as the internal energy $U$, i) find the temperature $T$ of this system. ii) Then invert your answer to get $U(T)$.

d) Find the heat capacity of this system.

e) Compare your results with those for spin-$1/2$ dipoles in a magnetic field, as treated in class and in PB §3.10.

2. Suppose $\Phi(x) = x^3$ and $X = d\Phi/dx$. Perform a Legendre transformation to find $\Psi(X)$ for $x \geq 0$.

3. Consider a semi-infinite 1D system for a single classical particle moving vertically ($z \geq 0$) with Hamiltonian $\mathcal{H}(p_z, z) = p_z^2 / 2m + A z^n$, $n > 0$

a) Write down the canonical distribution function $\rho(p_z, z)$ and show that it separates into the product of two factors, one dependent only on $p_z$ and the other only on $z$.

b) Find the mean potential energy $\langle A z^n \rangle$ ($z \geq 0$) of the particle ($z \geq 0$), and compare your result with the generalized equipartition derived in class for modes $\propto$ even powers of $z$ for $-\infty < z < \infty$.

Reminder: $\int_0^\infty \exp \left( -x^n \right) dx = \Gamma \left( 1 + \frac{1}{n} \right), \quad n > 0$

In-exam hint/reminder: if one knows $\int \exp[-a f(x)] dx$ one can find $\int f(x) \exp[-a f(x)] dx$ using the trick shown in class (hint: derivative).
4. [Adapted from a qualifier problem] Consider a low-density ideal gas of $N$ atoms confined to a container of volume $V$ and internal surface area $A$. While the interactions between atoms can be neglected, there is an attraction between atoms and the walls which must be taken into account. A simple model for the $N'$ atoms that are adsorbed onto the surface is to treat them as a two-dimensional (2D) classical ideal gas, where the energy of an adsorbed atom with 2D momentum $p$ is

$$\varepsilon(p) = \frac{|p|^2}{2m} - \varepsilon_0$$

(Treat $\varepsilon_0$ as some given positive binding energy.)

a) Show that the Helmholtz free energy of the $N'$ [indistinguishable] adsorbed atoms bound to the surface is

$$F(T,A) = -\beta^{-1} \left[ -N' \ln N' + N' + N' \ln A - 2 N' \ln \lambda_T + N' \beta \varepsilon_0 \right]$$

b) What is the chemical potential $\mu$, of the adsorbed atoms?

c) From your notes or P&B, recall that the chemical potential $\mu$ of the $N - N'$ atoms in the container is

$$\mu = k_B T \ln \left[ \frac{(N-N')\lambda_T^3}{V} \right]$$

When the atoms in the volume and those on the surface are in equilibrium with each other, what is the average number of atoms adsorbed as a function of temperature $T$?

d) Find the fraction of atoms sticking to the walls in the limits $T \to 0$ and $T \to \infty$, and briefly explain why these limits are reasonable.