Department of Physics
University of Maryland
College Park, MD 20742-4111

Physics 603 **HOMEWORK ASSIGNMENT #7** Spring 2013

Due date for problems on Thursday, April 25 [deadline on April 30].

1. a) PB 8.1:

   b) PB 8.2

2. PB 8.10a) and b). [Cf. 7.14 for Bose gases.]

3. a) Using results derived in class, show for Fermi gases that, to leading order in T, $S = (\pi^2/3)k_B{}^2Tg(\varepsilon_F)$

   b) Using the Sommerfeld low-temperature expansion as done in class, find $f_\nu(z)$ to order T^2. Be sure to eliminate μ from your answer, as done in class. Use your result to check the expansion for γ in PB problem 8.3, for κ_S in PB problem 8.4, and for χ in PB problem 8.14.

4, 5 . Qual problems (see next pages)

4.

Problem I.3

Consider a free-electron gas, with N electrons and dispersion relation $\epsilon = (\hbar|\mathbf{k}|)^2/2m$ in d dimensions, at temperature T=0.

(a) Show that the density of electronic states $G(\epsilon)$ satisfies

$$G(\epsilon) \propto \epsilon^{\alpha} \tag{1}$$

and find the value of α as a function of d=1,2,3. (Hint: the sum of these 3 values of α is 0.)

Recall:

$$G(\epsilon) \propto \int \frac{dS_\epsilon}{|\nabla_k \epsilon|} \tag{2}$$

where the integration in **k**-space is over a surface (sphere, circle, etc.) of constant energy ϵ. **[4 points]**

(b) **i.** What is the [physical] meaning of $G(\epsilon)\, d\epsilon$? **[1 point]**
ii. The proportionality (1) can be written as an equation in terms of a dimensionless constant A. Using dimensional arguments, show that

$$G(\epsilon) = \frac{AN\epsilon^{\alpha}}{{\epsilon_F}^{\alpha'}}, \tag{3}$$

where ϵ_F is the Fermi energy, and specify the relationship between α' and α. **[3 points]**
iii. Find the [d-dependent] value of A. **[2 points]**

For graphene (a single planar sheet of graphite), the electronic dispersion relation can be written as $\epsilon = \hbar v_s|\mathbf{k}|$ for small ϵ. (The points in **k**-space at which $\epsilon = 0$ are hence called Dirac points, and **k** is measured from them; you do not need this parenthetical information to do this problem.)

(c) **i.** Show that the density of states $G(\epsilon)$ is still proportional to $\epsilon^{\alpha}/{\epsilon_F}^{\alpha'}$, and find the new value of α. **[3 points]**
ii. Does the relationship between α' and α change from part (b)? If yes, how? If not, why not? **[1 point]**

(d) **i.** The low-temperature specific heat of a metal is known to be a power-law of T. Give a quick argument to show what this power is. (It may be helpful to draw a sketch of the change in the Fermi-Dirac distribution when T increases slightly from 0; use of the Sommerfeld expansion is not expected!) **[2 points]**
ii. For graphene with small doping, so that ϵ_F and $G(\epsilon_F)$ are non-zero, how does the temperature dependence of the low temperature specific heat compare with that found in part (d)-i? **[2 points]**

(e) In many cases a small gap (of size E_g) opens around $\epsilon = 0$. If ϵ_F lies inside this gap, write down the leading form of the temperature dependence of the low-T ($T \ll E_g/k_B$) specific heat. In the limit that $T \to 0$, what is the *ratio* of this specific heat to the specific heat in part (d)-ii? **[2 points]**

5.

First consider a classical ideal gas at temperature T consisting of N molecules and initially confined in a volume V_i. Then the gas is allowed to expand to a final volume V_f in two different ways:

(a) *Free expansion.* The gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum. Calculate the entropy change of the gas $\Delta S_{\rm irr}^{\rm gas} = S_f - S_i$ by comparing the number of accessible states before and after the expansion. (8 points)

(b) *Isothermal expansion.* The gas is in thermal contact with a reservoir of temperature T and experiences a slow reversible quasistatic expansion, e.g. produced by a slow motion of a piston that limits the gas volume. Calculate the work W done on the gas in this process, the change $\Delta U = U_f - U_i$ of the internal energy of the gas, and the heat Q transferred to the gas from the environment. Calculate the entropy change of the gas $\Delta S_{\rm rev}^{\rm gas} = S_f - S_i$ in this reversible process by using the formula $\Delta S = Q/T$. Compare your answers for $\Delta S_{\rm irr}^{\rm gas}$ and $\Delta S_{\rm rev}^{\rm gas}$. Are the two results the same or different? Explain why. (8 points).

(c) What are the entropy changes in the environment for these two cases: $\Delta S_{\rm irr}^{\rm env}$ and $\Delta S_{\rm rev}^{\rm env}$? What are the total entropy changes in the gas and the environment for these two cases: $\Delta S_{\rm irr}^{\rm tot} = \Delta S_{\rm irr}^{\rm gas} + \Delta S_{\rm irr}^{\rm env}$ and $\Delta S_{\rm rev}^{\rm tot} = \Delta S_{\rm rev}^{\rm gas} + \Delta S_{\rm rev}^{\rm env}$? Are $\Delta S_{\rm irr}^{\rm tot}$ and $\Delta S_{\rm rev}^{\rm tot}$ the same or different? Explain why. (5 points)

(d) Now consider a non-interacting degenerate Fermi gas made of N spin-1/2 fermions each of mass m and initially confined in a volume V_i at zero temperature $T = 0$. Calculate the Fermi momentum p_F, the Fermi energy E_F, and the energy per particle U/N. Express U/N in terms of E_F. (7 pts)

(e) This Fermi gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum to a final volume V_f. Assume that V_f is sufficiently large so that the Fermi gas becomes non-degenerate, i.e. classical. Calculate the final temperature T_f of the gas after the expansion. Express T_f in terms of E_F obtained above. (6 pts)

(f) Estimate by what factor V_f/V_i the initially degenerate Fermi gas needs to expand in order to become classical in the final state. (6 pts)
Hint: A gas behaves classically when the inter-particle separation $d = \sqrt[3]{N/V}$ is much greater than the thermal de Broglie length $\lambda = h/\sqrt{2\pi mkT}$, where h and k are the Planck and the Boltzmann constants.