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Physics 603

HOMEWORK ASSIGNMENT #8

Spring 2012

Due date for problems on Thursday, April 26 [deadline on May 1].

1. Using the Sommerfeld expansion with $\phi(\epsilon) = g(\epsilon)\varphi(\epsilon)$ show that to order $(k_B T/\epsilon_F)^2$, (sorry, no ϕ in the equation editor), where the lower limit of the integral is at or below the bottom of the band.

$$\int_{-\infty}^{\infty} g(\epsilon)\varphi(\epsilon)f(\epsilon)d\epsilon = \int_{-\infty}^{\epsilon_F} g(\epsilon)\varphi(\epsilon)d\epsilon + \frac{\pi^2}{6}(k_B T)^2 g(\epsilon_F)\varphi'(\epsilon_F)$$

2. a) PB 8.10a)

Consider an ideal Fermi gas, with energy spectrum $\epsilon \propto p^s$, contained in a box of “volume” V in a space of n dimensions. Show that, for this system,

(a) $PV = \frac{s}{n}U;$

b) PB 8.13, another Sommerfeld-expansion problem; use results from class to show:

Show that, *quite generally*, the low-temperature behavior of the chemical potential, the specific heat, and the entropy of an ideal Fermi gas is given by

$$\mu \simeq \epsilon_F \left[1 - \frac{\pi^2}{6} \left(\frac{\partial \ln a(\epsilon)}{\partial \ln \epsilon} \right)_{\epsilon=\epsilon_F} \left(\frac{kT}{\epsilon_F} \right)^2 \right],$$

and

$$C_V \simeq S \simeq \frac{\pi^2}{3} k^2 T a(\epsilon_F),$$

where $a(\epsilon)$ is the *density of (the single-particle) states* in the system. Examine these results for a gas with energy spectrum $\epsilon \propto p^s$, confined to a space of n dimensions, and discuss the special cases: $s = 1$ and 2 , with $n = 2$ and 3 .

This $a(\epsilon)$ is our $g(\epsilon)$.

3a) Consider a potential $u(\mathbf{r}) = \begin{matrix} \infty & \mathbf{r} < a \\ -\epsilon & a < \mathbf{r} < b \\ 0 & b < \mathbf{r} \end{matrix}$

$$-\epsilon \quad a < \mathbf{r} < b$$

$$0 \quad b < \mathbf{r}$$

Find the second virial coefficient $B_2(T)$ and its limits at high T and low T (relative to ϵ).

b) In *one dimension* (in contrast to three dimensions in part a) and with $\epsilon = 0$ (so that this is a “Tonks gas” of hard rods of length a), evaluate $B_2(T)$ and $B_3(T)$.

4, 5 . Qual problems (see next pages)

4.

Problem I.3

Consider a free-electron gas, with N electrons and dispersion relation $\epsilon = (\hbar|\mathbf{k}|)^2/2m$ in d dimensions, at temperature $T=0$.

- (a) Show that the density of electronic states $G(\epsilon)$ satisfies

$$G(\epsilon) \propto \epsilon^\alpha \quad (1)$$

and find the value of α as a function of $d=1,2,3$. (Hint: the sum of these 3 values of α is 0.)

Recall:

$$G(\epsilon) \propto \int \frac{dS_\epsilon}{|\nabla_{\mathbf{k}} \epsilon|} \quad (2)$$

where the integration in \mathbf{k} -space is over a surface (sphere, circle, etc.) of constant energy ϵ . [4 points]

- (b) i. What is the [physical] meaning of $G(\epsilon) d\epsilon$? [1 point]
 ii. The proportionality (1) can be written as an equation in terms of a dimensionless constant A . Using dimensional arguments, show that

$$G(\epsilon) = \frac{AN\epsilon^\alpha}{\epsilon_F^{\alpha'}}, \quad (3)$$

where ϵ_F is the Fermi energy, and specify the relationship between α' and α . [3 points]

- iii. Find the [d -dependent] value of A . [2 points]

For graphene (a single planar sheet of graphite), the electronic dispersion relation can be written as $\epsilon = \hbar v_s |\mathbf{k}|$ for small ϵ . (The points in \mathbf{k} -space at which $\epsilon = 0$ are hence called Dirac points, and \mathbf{k} is measured from them; you do not need this parenthetical information to do this problem.)

- (c) i. Show that the density of states $G(\epsilon)$ is still proportional to $\epsilon^\alpha/\epsilon_F^{\alpha'}$, and find the new value of α . [3 points]
 ii. Does the relationship between α' and α change from part (b)? If yes, how? If not, why not? [1 point]
- (d) i. The low-temperature specific heat of a metal is known to be a power-law of T . Give a quick argument to show what this power is. (It may be helpful to draw a sketch of the change in the Fermi-Dirac distribution when T increases slightly from 0; use of the Sommerfeld expansion is not expected!) [2 points]
 ii. For graphene with small doping, so that ϵ_F and $G(\epsilon_F)$ are non-zero, how does the temperature dependence of the low temperature specific heat compare with that found in part (d)-i? [2 points]
- (e) In many cases a small gap (of size E_g) opens around $\epsilon = 0$. If ϵ_F lies inside this gap, write down the leading form of the temperature dependence of the low- T ($T \ll E_g/k_B$) specific heat. In the limit that $T \rightarrow 0$, what is the *ratio* of this specific heat to the specific heat in part (d)-ii? [2 points]

5.

Problem I.3

Qualifying Examination

January 2011

First consider a classical ideal gas at temperature T consisting of N molecules and initially confined in a volume V_i . Then the gas is allowed to expand to a final volume V_f in two different ways:

- (a) *Free expansion.* The gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum. Calculate the entropy change of the gas $\Delta S_{\text{irr}}^{\text{gas}} = S_f - S_i$ by comparing the number of accessible states before and after the expansion. (8 points)
- (b) *Isothermal expansion.* The gas is in thermal contact with a reservoir of temperature T and experiences a slow reversible quasistatic expansion, e.g. produced by a slow motion of a piston that limits the gas volume. Calculate the work W done on the gas in this process, the change $\Delta U = U_f - U_i$ of the internal energy of the gas, and the heat Q transferred to the gas from the environment. Calculate the entropy change of the gas $\Delta S_{\text{rev}}^{\text{gas}} = S_f - S_i$ in this reversible process by using the formula $\Delta S = Q/T$. Compare your answers for $\Delta S_{\text{irr}}^{\text{gas}}$ and $\Delta S_{\text{rev}}^{\text{gas}}$. Are the two results the same or different? Explain why. (8 points).
- (c) What are the entropy changes in the environment for these two cases: $\Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{env}}$? What are the total entropy changes in the gas and the environment for these two cases: $\Delta S_{\text{irr}}^{\text{tot}} = \Delta S_{\text{irr}}^{\text{gas}} + \Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{tot}} = \Delta S_{\text{rev}}^{\text{gas}} + \Delta S_{\text{rev}}^{\text{env}}$? Are $\Delta S_{\text{irr}}^{\text{tot}}$ and $\Delta S_{\text{rev}}^{\text{tot}}$ the same or different? Explain why. (5 points)
- (d) Now consider a non-interacting degenerate Fermi gas made of N spin-1/2 fermions each of mass m and initially confined in a volume V_i at zero temperature $T = 0$. Calculate the Fermi momentum p_F , the Fermi energy E_F , and the energy per particle U/N . Express U/N in terms of E_F . (7 pts)
- (e) This Fermi gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum to a final volume V_f . Assume that V_f is sufficiently large so that the Fermi gas becomes non-degenerate, i.e. classical. Calculate the final temperature T_f of the gas after the expansion. Express T_f in terms of E_F obtained above. (6 pts)
- (f) Estimate by what factor V_f/V_i the initially degenerate Fermi gas needs to expand in order to become classical in the final state. (6 pts)
Hint: A gas behaves classically when the inter-particle separation $d = \sqrt[3]{N/V}$ is much greater than the thermal de Broglie length $\lambda = h/\sqrt{2\pi mkT}$, where h and k are the Planck and the Boltzmann constants.