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Physics 603 **HOMEWORK ASSIGNMENT #4** Spring 2012

Due date for problems on Thursday, March 1 [deadline on March 6].

1. Construct the normalized canonical distribution $\rho(\mathbf{p},\mathbf{r})$ for a single (classical) particle moving in the presence of gravity in a vertical cylinder of [large] cross-sectional area A (and infinite height above its base at z=0). Then calculate

a) the probability of the particle being found between heights h and h+dh above the cylinder's base,

b) the mean height, and

c) the mean (total) energy of the particle.

2. A particle of mass m moves in a circle of radius R in a vertical plane in the earth's gravitational field (so forming an ideal rigid pendulum).

a) Write the Hamiltonian of the system in terms of the angular coordinate θ description the displacement from the position of lowest potential energy.

b) Then construct the normalized canonical distribution function $\rho(\theta, \ell=p_\theta)$.

c) Obtain therefrom a formula for the probability of an angular displacement between θ and $\theta + d\theta$.

d) In the limit of large mass (or low temperature), find an explicit expression for the mean square angular displacement $\langle\theta^2\rangle$ for the mean height above the lowest point of the circle. [Use the small-angle approximation $\cos\theta \rightarrow 1 - ½\,\theta^2$].

e) Find a similar expression for $\langle\theta^2\rangle$ at high temperatures, both the leading term and the first T-dependent correction.

Problems 3, 4, and 5, on the subsequent pages, are taken from qualifier exams.

3. First consider a classical ideal gas at temperature T consisting of N molecules and initially confined in a volume V_i. Then the gas is allowed to expand to a final volume V_f in two different ways:

(a) *Free expansion.* The gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum. Calculate the entropy change of the gas $\Delta S_{\text{irr}}^{\text{gas}} = S_f - S_i$ by comparing the number of accessible states before and after the expansion. (8 points)

(b) *Isothermal expansion.* The gas is in thermal contact with a reservoir of temperature T and experiences a slow reversible quasistatic expansion, e.g. produced by a slow motion of a piston that limits the gas volume. Calculate the work W done on the gas in this process, the change $\Delta U = U_f - U_i$ of the internal energy of the gas, and the heat Q transferred to the gas from the environment. Calculate the entropy change of the gas $\Delta S_{\text{rev}}^{\text{gas}} = S_f - S_i$ in this reversible process by using the formula $\Delta S = Q/T$. Compare your answers for $\Delta S_{\text{irr}}^{\text{gas}}$ and $\Delta S_{\text{rev}}^{\text{gas}}$. Are the two results the same or different? Explain why. (8 points).

(c) What are the entropy changes in the environment for these two cases: $\Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{env}}$? What are the total entropy changes in the gas and the environment for these two cases: $\Delta S_{\text{irr}}^{\text{tot}} = \Delta S_{\text{irr}}^{\text{gas}} + \Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{tot}} = \Delta S_{\text{rev}}^{\text{gas}} + \Delta S_{\text{rev}}^{\text{env}}$? Are $\Delta S_{\text{irr}}^{\text{tot}}$ and $\Delta S_{\text{rev}}^{\text{tot}}$ the same or different? Explain why. (5 points) [The second half of this problem deals with a degenerate Fermi gas.]

4. "Whether the second law really is a law in any meaningful and scientific sense of the term may legitimately be doubted. Interestingly enough, the theory of evolution itself contradicts the second law of thermodynamics, as does each and every instance of life." (Tom Bethell, *The American Spectator*, Nov. 1980)

(a) Give a quantitative statement of the second law of thermodynamics involving the entropy, clearly stating under what conditions the law holds. [4 points]

(b) Comment briefly (in fifty words or, preferably, fewer) on how life might exist and evolution might occur without violating the second law. [4 points]

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(c) Lord Kelvin's classical statement of the second law was that a transformation whose only final result is to transform heat (from a fixed-temperature reservoir) into work is impossible. Show that this is true by using part (a) and energy conservation. [4 points]

(d) By having two reservoirs at T_h and T_l, where $T_l < T_h$, it is possible to convert some heat into work. Derive an expression for the efficiency of a reversible motor which operates between T_l and T_h, in terms of T_l and T_h. [4 points]

(e) Calculate the change in entropy of an ideal monatomic gas that is reversibly taken from a temperature T_l to a temperature T_h. Assume that the volume of the gas is constant. [4 points]

5. UMD qualifier problem, January 2004: Probability distribution of money.

This problem explores an analogy between the Boltzmann-Gibbs probability distribution of energy in statistical physics and the probability distribution of money in a closed system of economic agents. Consider a system consisting of $N \gg 1$ economic agents. At a given moment of time, each agent i has a non-negative amount of money $m_i \geq 0$ (debt is not permitted). As the agents engage in economic activity, money is constantly transferred between the agents in the form of payments. However, the total amount of money is conserved in binary transactions between agents: $m_i + m_j = m_i' + m_j'$. This condition is analogous to conservation of energy in collisions between atoms in a gas. We also assume that the system is closed, so the total amount of money in the system M is also conserved.

We wish to obtain a formula for the money distribution function $P(m)$ for the system in statistical equilibrium. It is defined so that $P(m)\,dm$ is the fraction of agents with money in the interval $[m, m+dm]$ and satisfies the standard normalization conditions:

$$\int_0^\infty P(m)\,dm = 1, \qquad \int_0^\infty m\,P(m)\,dm = \langle m \rangle = M/N, \tag{1}$$

where $\langle m \rangle = M/N$ is the average amount of money per agent.

(a) Let us divide the money semi-axis $m \geq 0$ into equal intervals Δm and count the number of agents belonging to each interval: N_1, N_2, $N_3, \ldots$ Obviously $\sum_{r=1}^{\infty} N_r = N$. This configuration can be realized in many different ways by moving agents between the intervals while preserving the occupation numbers N_1, N_2, $N_3, \ldots$ Write a combinatorial formula for the number of ways W a given distribution N_1, N_2, $N_3, \ldots$ can be realized.

(b) Using the Stirling approximate formula $\ln n! \approx n \ln n - n$ for $n \gg 1$, obtain an expression for $S = \ln W$, the entropy of the distribution.

(c) Using the method of Lagrange multipliers, find the configuration N_r^* that maximizes entropy S under constraints that the total number of agents is conserved and the total amount of money is conserved:

$$\sum_{r=1}^{\infty} N_r = N, \qquad \sum_{r=1}^{\infty} m_r N_r = M. \tag{2}$$

(d) Obtain the money distribution function $P(m)$ for the fraction of agents belonging to a given interval Δm: $P(m_r)\,\Delta m = N_r^*/N$. Determine the values of the Lagrange multipliers from the normalization conditions (1).

(e) Compare the obtained result for $P(m)$ with the Boltzmann-Gibbs formula for the probability distribution of energy $P(E)$ in physics. What is the analog of temperature in the system of economic agents? Compare with $\langle m \rangle$.