Physics 601
Dr. Dragt
Fall 2002

Reading Assignment #9:

1. Dragt
   (a) By this time you should have read everything about rotations.
2. Symon
   (a) Excerpt on MOVING COORDINATE SYSTEMS.
3. Goldstein
   (a) Sections 5.5 through 5.9 of Chapter 5. By this time you should have read all of Chapters 1 through 5.

Problem Set 9 due Wednesday, 11/20/02

76. The Levi-Civita symbol (tensor density) ϵ_{ijk} is defined by the rules $\epsilon_{123} = 1$, $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$.
    (a) Show that $\epsilon_{kji} = -\epsilon_{ijk}$. Thus ϵ_{ijk} is *completely* antisymmetric.
    (b) List all nonzero values of ϵ_{ijk}.
    (c) If $\boldsymbol{a} = \sum_i a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = \sum_j b_j \boldsymbol{e}_j$, show that $\boldsymbol{a} \times \boldsymbol{b} = \sum_{ijk} \epsilon_{ijk} a_i b_j \boldsymbol{e}_k$.
    (d) Show that $\boldsymbol{e}_i \times \boldsymbol{e}_j = \sum_k \epsilon_{ijk} \boldsymbol{e}_k$.
    (e) Evaluate: $\sum_k \epsilon_{ijk}\epsilon_{\ell mk} = ?$ Use your result to prove the vector identity
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}).$$

(f) Show that $(J_i)_{jk} = -\epsilon_{ijk}$.

77. Suppose A, B, and C are $n \times n$ matrices and suppose a Lie product is defined by the commutator rule

$$[A, B] = AB - BA.$$

Verify that the Jacobi identity,

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0,$$

is satisfied.

78. Verify the relations:

(a) $\operatorname{tr}(\sigma_j) = 0$ for $j = 1$ to 3.

(b) $\operatorname{tr}(\sigma_j \sigma_k) = 2\delta_{jk}$ for $j, k = 1$ to 4.

(c) $(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = (\boldsymbol{a} \cdot \boldsymbol{b})\sigma_0 + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{\sigma}$

(d) $[\mathcal{J}_j, \mathcal{J}_k] = \sum_{\ell} \epsilon_{jk\ell} \mathcal{J}_\ell$ with $\mathcal{J}_j = -(i/2)\sigma_j$.

79. See the discussion after Theorem 44.

(a) Find the eigenvalues and eigenvectors of J_3 and $\mathcal{J}_3 = -\frac{1}{2}i\ \sigma_3$.

(b) Find the eigenvalues and eigenvectors of $J^2 = \sum_1^3 J_j^2$ and $\mathcal{J}^2 = \sum_1^3 \mathcal{J}_j^2$.

(c) Find the eigenvalues and eigenvectors of $R(\boldsymbol{e}_3, \theta)$ and $u(\boldsymbol{e}_3, \theta)$.

80. Refer to Theorem 35.

(a) Show that $M(R; \alpha) = M(I, \alpha)M(R, 0)$ can be written in exponential form, at least near the identity. (It is actually possible globally.)

(b) Study the Lie Algebra for the Euclidean Group. That is, exhibit a convenient basis and compute the structure constants.

81. Let $R_1(\boldsymbol{n}_1; \theta_1)$ and $R_2(\boldsymbol{n}_2; \theta_2)$ be two given rotations. Compute the $\boldsymbol{n}$ and θ for $R(\boldsymbol{n}; \theta) = R_1 R_2$. Hint: Use the 2×2 representation. State your results in terms of the vectors $\boldsymbol{\tau} = \boldsymbol{n} \tan \frac{\theta}{2}$, etc.

82. This problem relates three different parameterizations of the Rotation Group.

    (a) Given $\boldsymbol{n}$ and θ for $R(\boldsymbol{n}; \theta)$, compute the quaternion 4-vector w such that $R(w) = R(\boldsymbol{n}; \theta)$.

    (b) Given the Euler angles ϕ, θ, ψ for $R(\phi, \theta, \psi)$, compute w such that $R(w) = R(\phi, \theta, \psi)$.

    (c) Given the Euler angles ϕ, θ, ψ for $R(\phi, \theta, \psi)$, compute $\boldsymbol{n}$ and χ such that $R(\boldsymbol{n}; \chi) = R(\phi, \theta, \psi)$.

Problems 81 and 82 are worth 15 points. All others are worth 10 points.

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