Physics 601
Dr. Dragt
Fall 2002

Reading Assignment #6:

1. Dragt
   (a) The Rotation Group
2. Goldstein
   (a) Sections 4.1 through 4.8 of Chapter 4.

Problem Set 6 due Monday, 10/28/02

54. Assume that the earth is in a circular orbit around the sun, and that $M_s \gg M_e$. Imagine that the sun's mass is suddenly reduced to 1/2 its previous value. What happens to the earth's orbit? Hint: Use the Virial Theorem.

55. Find the most general spherically symmetric potential such that if $r = h(\theta)$ is an orbit, so is $r = \kappa h(\theta)$ for every positive constant κ.

56. Find the most general potential which is spherically symmetric, corresponding to an attractive force everywhere, and such that for every bounded orbit $r = h(\theta)$ we have $h(\theta + \theta_0) = h(\theta)$. Here the constant θ_0 is not to depend on the orbit, but is to be the same for all orbits. In essence, this problem calls for a recitation of the proof of Bertrand's theorem.

57. Derive Kepler's equation relating the mean anomaly to the eccentric anomaly. [See item B2 (page 5) of Berkeley Mechanics Notes.] Show that Kepler's equation
$$\alpha = \beta - \epsilon \sin \beta$$

can be solved for β in terms of α to give the result

$$\beta = \alpha + 2\sum_{n=1}^{\infty} n^{-1} J_n(n\epsilon) \sin n\alpha$$

where the J_n are Bessel functions. Incidentally, Bessel functions were used here (by Bessel) before they were used in Electricity and Magnetism. Also note, that despite the inference apparently implied in problem G3.2, the solution is *not* formal if the functions J_n are assumed to be known. That is, the series is perfectly well behaved, and even absolutely convergent for $|\epsilon| < 1$ and α real. Finally, note that G3.2 contains a misprint in an equation.

58. Consider motion in a central force field described by $V(r) = \lambda/r^2$.

    (a) Show that circular orbits (in the attractive case) are unstable.

    (b) Find $r(t)$, $\theta(t)$, and $r(\theta)$ for the general case (arbitrary initial conditions and λ either positive or negative). Sketch various possible orbits.

59. Find the orbit of a planet in the General Theory of Relativity, from the Lagrangian given in item C2 (page 8) of the Berkeley Mechanics Notes. First show that the Schwarzschild metric for a spherical gravitating body given by

$$ds^2 = c^2(1 - a/r)dt^2 - (1 - a/r)^{-1}dr^2 - r^2[d\theta^2 + \sin^2\theta d\phi^2]$$

leads to the advertised Lagrangian. Next show that the motion is in a plane; take this plane to be the plane $\theta = \pi/2$. Introduce $u = 1/r$ and show that the equation for the orbit may be written:

$$\left(\frac{du}{d\phi}\right)^2 + u^2(1 - au) = \frac{c^2}{L^2}(2E/c^2 + au)$$

where L and E are constants; comparing with (1) in item A1 we may identify these with the angular momentum and total energy, provided the test-particle is of unit mass.

In actual planetary motion au is always small compared to unity, and the equations may be solved by approximation. Evaluate au for Mercury and the Earth, i.e. produce a number!

Solve the equation for the orbit for the case of an almost circular orbit and compute the precession of the perihelion of Mercury and of the Earth in seconds of arc per century.

You are expected to carry the numerical work to its conclusion. The following data will be needed:

One light-year $= 5.88 \times 10^{12}$ miles
Radius of orbit of Earth $= 93,000,000$ miles $= R_\oplus$
Radius of orbit of Mercury $= 0.39 R_\oplus$

The answer for Mercury is ~ 40 seconds of arc/century.

60. D1.4.6

61. As a simple application of the variational equation formalism, consider the motion of a particle of mass m in the exterior gravitational field of a heavy sphere of mass M. Assume that $M \gg m$. Then one can work in a frame where M is fixed at the origin. Furthermore, since the force is central, one can assume that motion takes plane in a plane. Finally, let G denote the gravitational constant as measured by Hank Cavendish.

    (a) Using plane polar coordinates r and ϕ, write the Lagrangian for the particle of mass m.

    (b) Find the equations of motion for r and ϕ.

    (c) Show that there is a circular orbit solution of the form

    $$r = R\ ,\ \phi = \omega t, \tag{a, b}$$

    and give a relation which determines ω.

    (d) Consider variational solutions near the circular orbit by writing

    $$r = R + \epsilon\rho \tag{c}$$

    $$\phi = \omega t + \epsilon\psi. \tag{d}$$

    Find the variational equations obeyed by ρ and ψ. Hint: Rather than using the full formal machinery, it may be easier to substitute (c,d) into the equations of motion and then retain only low order terms.

(e) During his walk in outer space while in circular orbit about the earth, the cosmonaut A. Leonov faced the earth and threw the lens cap of his movie camera directly *toward* the earth with velocity v. Suppose he threw the lens cap at the moment $t = 0$ when $\phi = 0$. Then, at this moment, one may assume the initial conditions

$$\epsilon\rho = 0 \ , \ \epsilon\dot{\rho} = -v, \qquad \text{(e, f)}$$

$$\epsilon\psi = 0 \ , \ \epsilon\dot{\psi} = 0. \qquad \text{(g, h)}$$

Find the variational solutions $\rho(t)$ and $\psi(t)$ corresponding to these initial conditions. In particular, show that if comrade Leonov subsequently turns himself to face *away* from the earth, then he will see the lens cap coming toward him one orbital period later and, to the accuracy of the variational solution, he should be able to catch the lens cap!

(f) Find the variational solutions $\rho(t)$ and $\psi(t)$ for general initial conditions in the plane of the orbit. Suppose that the lens cap is thrown out in the direction of the orbit at $t = 0$. Then the lens cap is going faster than the cosmonaut and his space craft. Show that the lens cap is found *behind* the cosmonaut one period later even though it is going *faster*. Conversely, show that if the lens cap is thrown against the orbit velocity, so that it is going more slowly than the cosmonaut, he will find it ahead of him one period later.

(g) What happens if the lens cap is thrown out of the plane of the orbit?