Physics 601
Dr. Dragt
Fall 2002

Reading Assignment #5:

1. Dragt
   (a) Section 1.3, Hamiltonian formulation of light optics (in Chapter 1, THE UBIQUITY OF LAGRANGIAN AND HAMILTONIAN DYNAMICS). Sections 1.2 and 1.3 of this chapter are equivalent to sections 1.5 and 1.6 of Chapter 1 (Introductory Concepts) that you should have already read.
   (b) NOTES ON VECTORS AND VECTOR SPACES.
   (c) MORE NOTES ON MATRICES.
2. Goldstein
   (a) Make sure that you have read all the parts of Goldstein that have already been assigned.

Problem Set 5 due Monday, 10/21/02

44. D1.4.2 [problem 4.2 in Chapter 1 (Introductory Concepts) of Dragt].
45. Consider the magnetic field produced by an infinitely long straight wire running along the z-axis and carrying a current I.

(a) Using cylindrical coordinates $\rho\ \phi\ z$ (why?), find a suitable vector potential for the magnetic field.

(b) Again using cylindrical coordinates, write the Lagrangian for a particle of mass m and charge q moving in this field.

(c) Find the Hamiltonian.

(d) Find *three* constants of motion, and reduce the solution of the equations of motion to quadratures.

(e) Sketch a low energy orbit.

46. Suppose that Fermat and Taylor were assigned the task of surveying the Shara desert using mirrors, a fast timing device, and a pulsed laser beam. Using his keen powers of observation, Fermat would notice that the floor of the desert and the air just above it were very hot. Hence the density of the air near the desert floor would be lower; and its index of refraction would be a little bit less than that of the air higher up.

They would both observe that, to shine light from point A to point B, it was necessary to point the laser beam a little bit downwards. Using Fermat's principle, calculate the *approximate* path of the beam. Taylor would suggest that you use his series and write for the index of refraction $n(y) \simeq n(h) + n'(h)(y - h)$. The quantity $n'(h)$ is very small. He would thus also suggest that you write

$$\sqrt{1 + (dy/dx)^2} \simeq 1 + \frac{1}{2}(dy/dx)^2$$

since the path of the laser beam is nearly horizontal. In the same spirit he would neglect the cross term $n'(h)(dy/dx)^2$.

47. Consider the problem of tracing light rays through an optical medium characterized by a position dependent index of refraction $n(\boldsymbol{r})$. Suppose rays are traced using z as an independent variable so that each ray is characterized by the two functions $x(z)$ and $y(z)$.

   (a) Show that an effective Lagrangian for the rays is given by the expression

   $$L(x, y; x', y'; z) = n(x, y, z)[1 + (x')^2 + (y')^2]^{1/2}$$

   where a $'$ denotes the effect of the operator d/dz.

   (b) Find the momenta p_x and p_y canonically conjugate to the coordinates x and y.

   (c) Find the Hamiltonian $H(x, y; p_x, p_y; z)$.

   (d) Consider the case of light rays incident on a sheet of glass which is normal to the z axis. See the picture below.

   In this case, around the interface $z = d$, $n(\boldsymbol{r})$ can be written in the form

   $$n(\boldsymbol{r}) = n_1\theta(d - z) + n_2\theta(z - d)$$

where θ is the Heavyside step function. Show from the Hamiltonian equations of motion that p_x and p_y for any incident ray are continuous across the interface. From this observation derive Snell's law of refraction.

(e) Fiber optics is now commonly used in optical communications systems to trasmit laser light signals over large distances. A Selfoc (self focussing) fiber whose axis is along the z axis is characterized by the position dependent refractive index given by the expression

$$n^2(x, y, z) = n_0^2[1 - \alpha^2(x^2 + y^2)]$$

where n_0 and α are constants. Find the functions $x(z)$ and $y(z)$ for a general light ray in a Selfoc fiber.

48. Let $z_1 \cdots z_m$ be m variables and consider the differential form $\Sigma C_a(z)dz_a$ where $C_1 \cdots C_m$ are m specified functions of the variables z.

(a) Suppose there is a function $f(z)$ such that

$$df = \sum_{a=1}^{m} C_a(z)dz_a. \tag{i}$$

Then the differential form is said to be *perfect* or *exact*. Show that in this case the functions C satisfy the property

$$\partial C_a/\partial z_b - \partial C_b/\partial z_a = 0. \tag{ii}$$

(b) Suppose the functions C do satisfy the property (ii) in some simply connected region R. Then show that within R there is a well-defined function f such that (i) is true.

Hint: Consider all possible paths in the region R. Let $z(\tau)$ be such a path parameterized by a variable τ. Next consider the functional $F[z(\tau)]$ over paths defined by

$$F[z(\tau)] = \int_{\tau^1}^{\tau^2} L(z, \dot{z})d\tau. \tag{iii}$$

Where L denotes the quantity

$$L(z, \dot{z}) = \Sigma C_a(z)(dz_a/d\tau). \tag{iv}$$

Use the calculus of variations and (ii) to show that the functional F depends only on the end points of the path, and not on the actual choice of path joining the two end points. Now use this result to construct f.

(c) Show that the integral (iii) is path (but not end point) independent if and only if (ii) is true.

(d) Relate the results of parts a, b, and c to Stokes' theorem and the existence of conservative vector fields in the case $m = 3$.

49. G2.3 (problem 3 in Chapter 2 of Goldstein).

50. G2.4

51. G2.1

52. G2.17

53. G2.24