Theoretical Dynamics September 16, 2010

Homework 2

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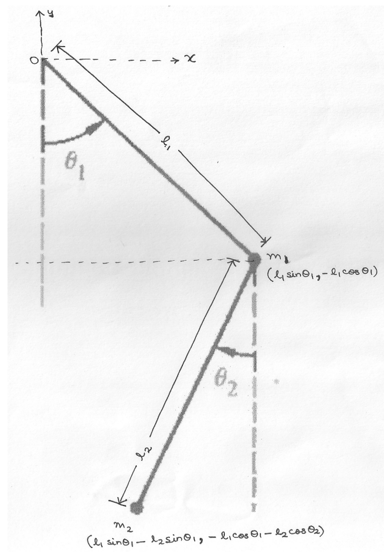

Taking the point of support as the origin and the axes as shown, the coordinates are

$$(x_1, y_1) = (l_1 \sin\theta_1, -l_1 \cos\theta_1) \tag{1}$$
$$(x_2, y_2) = (l_1 \sin\theta_1 - l_2 \sin\theta_2, -l_1 \cos\theta_1 - l_2 \cos\theta_2) \tag{2}$$

The Lagrangian is

$$L = T - V \tag{3}$$

where

$$\begin{aligned} T &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 - 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2)) \end{aligned} \tag{4}$$

and

$$V = -m_1 g l_1 \cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \tag{5}$$

So,

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 + \theta_2) + m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \tag{6}$$

The derivatives are

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1+\theta_2), \qquad \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1+\theta_2) \tag{7}$$

$$\frac{\partial L}{\partial \theta_1} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1+\theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1, \qquad \frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1+\theta_2) - m_2 g l_2 \sin\theta_2 \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1+m_2)l_1^2\ddot{\theta}_1 - m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1+\theta_2) + m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1+\theta_2) + m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \tag{10}$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0 \end{aligned}$$

that is,

$$(m_1+m_2)l_1^2\ddot{\theta}_1 - m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1+\theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1+\theta_2) + (m_1+m_2) g l_1 \sin\theta_1 = 0 \tag{11}$$

$$m_2 l_1^2 \ddot{\theta}_2 - m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1+\theta_2) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1+\theta_2) + m_2 g l_2 \sin\theta_2 = 0 \tag{12}$$

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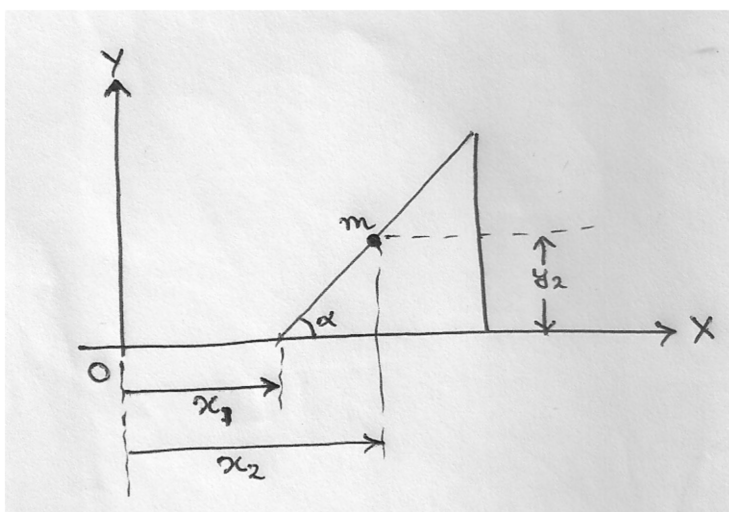

$$\begin{aligned} \text{Kinetic Energy} \quad T &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2+\dot{y}_2^2) && (13) \\ \text{Potential Energy} \quad V &= mgy_2 && (14) \end{aligned}$$

Constraint:

$$G(x_1, x_2, y_2) = y_2 - (x_2 - x_1)\tan\alpha = 0 \tag{15}$$

Lagrangian:

$$L = T - V = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) - mgy_2 \tag{16}$$

Constrained Lagrangian:

$$L_c = T - V - \lambda G = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) - mgy_2 - \lambda[y_2 - (x_2 - x_1)\tan\alpha] \tag{17}$$

The Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{19}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right) - \frac{\partial L}{\partial y_2} = 0 \tag{20}$$

give

$$M\ddot{x}_1 + \lambda\tan\alpha = 0 \tag{21}$$

$$m\ddot{x}_2 - \lambda\tan\alpha = 0 \tag{22}$$

$$m\ddot{y}_2 + mg + \lambda = 0 \tag{23}$$

Adding (21) and (22) we get

$$M\ddot{x}_1 + m\ddot{x}_2 = 0 \tag{24}$$

which upon one integration wrt time, yields the expected result that the l**inear momentum of the (block + wedge) system in the X-direction is constant**. Multiplying (23) throughout by $\tan\alpha$, using (15) to write $\ddot{y}_2 = (\ddot{x}_2 - \ddot{x}_1)\tan\alpha$ and substituing $\lambda\tan\alpha = -M\ddot{x}_1$ from (21) we get

$$\begin{aligned} m(\ddot{x}_2 - \ddot{x}_1)\tan\alpha + mg + \lambda &= 0 \\ \implies -(M+m)\ddot{x}_1\tan^2\alpha + mg\tan\alpha - M\ddot{x}_1 &= 0 \end{aligned}$$

So,

$$\ddot{x}_1 = \frac{m}{M}\frac{g\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{25}$$

$$\ddot{x}_2 = -\frac{g\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{26}$$

$$\ddot{y}_2 = -\left(1+\frac{m}{M}\right)\frac{g\tan^2\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{27}$$

$$\lambda = -\frac{mg}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{28}$$

The signs are consistent: as the particle descends the slope of the wedge, it moves to the left in the 'lab' frame, as the wedge moves to the right, conserving linear momentum in the horizontal direction. Also, as $m/M \to 0$, we recover the solution for a particle moving down a stationary wedge: $\ddot{x}_1 = 0$, $\ddot{x}_2 = -g\sin\alpha\cos\alpha$, $\ddot{y}_2 = -g\sin^2\alpha$ (so that the acceleration of the particle along the incline is $\sqrt{\ddot{x}_2^2 + \ddot{y}_2^2} = g\sin\alpha$).

Work done by the constraint forces

The three 'constraint forces' are

$$F_{x_1} = \lambda \frac{\partial G}{\partial x_1} = \lambda \tan\alpha = -\frac{mg\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{29}$$

$$F_{x_2} = \lambda \frac{\partial G}{\partial x_2} = -\lambda \tan\alpha = \frac{mg\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{30}$$

$$F_{y_2} = \lambda \frac{\partial G}{\partial y_2} = \lambda = -\frac{mg}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1} \tag{31}$$

The accelerations found above are constant, so the velocity varies linearly with time. Assuming that at $t = 0$, the wedge and particle both have zero velocity, the work done by the constraint force on the wedge is

$$\begin{aligned} W_1 &= \int F_{x_1} dx_1 \\ &= \frac{1}{2} F_{x_1} \ddot{x}_1 t^2 \\ &= \frac{1}{2}\left(-\frac{mg\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right)\left(\frac{m}{M}\frac{g\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right) t^2 \qquad (32) \\ &= -\frac{1}{2}\frac{\frac{m^2}{M} g^2 \tan^2\alpha}{\left[\left(1+\frac{m}{M}\right)\tan^2\alpha + 1\right]^2} t^2 \qquad (33) \end{aligned}$$

Similarly, the work done by the constraint force on the particle is

$$\begin{aligned} W_2 &= \int F_{x_2} dx_2 + \int F_{y_2} dy_2 \\ &= \frac{1}{2} F_{x_2} \ddot{x}_2 t^2 + \frac{1}{2} F_{y_2} \ddot{y}_2 t^2 \\ &= \frac{1}{2}\left(\frac{mg\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right)\left(-\frac{g\tan\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right) t^2 \\ &\quad + \frac{1}{2}\left(-\frac{mg}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right)\left(-\left(1+\frac{m}{M}\right)\frac{g\tan^2\alpha}{\left(1+\frac{m}{M}\right)\tan^2\alpha + 1}\right) t^2 \\ &= -\frac{1}{2}\frac{mg^2\tan^2\alpha}{\left[\left(1+\frac{m}{M}\right)\tan^2\alpha + 1\right]^2} t^2 + \frac{1}{2}\frac{m\left(1+\frac{m}{M}\right) g^2\tan^2\alpha}{\left[\left(1+\frac{m}{M}\right)\tan^2\alpha + 1\right]^2} t^2 \\ &= \frac{1}{2}\frac{\frac{m^2}{M} g^2 \tan^2\alpha}{\left[\left(1+\frac{m}{M}\right)\tan^2\alpha + 1\right]^2} t^2 \qquad (34) \end{aligned}$$

We note that $W_1 + W_2 = 0$, confirming the fact that the total work done on the system by the constraint forces in time t is zero. This is consistent with the fact that the constraint forces are internal to the system, and the constraint $G = 0$ is independent of time.

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The given Lagrangian density is

$$\mathcal{L} = \frac{h^2}{8\pi^2 m}\nabla\psi \cdot \nabla\psi^* + V\psi^*\psi + \frac{h}{4\pi i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*) \tag{35}$$

The Euler-Lagrange equation for ψ is

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{36}$$

that is,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla\psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{37}$$

The derivatives are

$$\begin{aligned}
\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= \frac{h}{4\pi i}\psi^* \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) &= \frac{h}{4\pi i}\dot{\psi}^* \\
\frac{\partial \mathcal{L}}{\partial(\nabla\psi)} &= \frac{h^2}{8\pi^2 m}\nabla\psi^* \\
\nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla\psi)}\right) &= \frac{h^2}{8\pi^2 m}\nabla^2\psi^* \\
\frac{\partial \mathcal{L}}{\partial \psi} &= V\psi^* - \frac{h}{4\pi i}\dot{\psi}^*
\end{aligned}$$

Substituting into (37), we get

$$\frac{h}{4\pi i}\dot{\psi}^* + \frac{h^2}{8\pi^2 m}\nabla^2\psi^* - V\psi^* + \frac{h}{4\pi i}\dot{\psi}^* = 0 \tag{38}$$

or

$$\frac{ih}{2\pi}\frac{d\psi}{dt} = -\frac{h^2}{8\pi^2 m}\nabla^2\psi + V\psi \tag{39}$$

which is Schrodinger's equation. The momentum canonically conjugate to ψ is

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{h}{4\pi i}\psi^* \tag{40}$$

So, the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \Pi\dot{\psi} + \Pi^*\dot{\psi}^* - \mathcal{L} & (41)\\
&= \frac{h}{4\pi i}\psi^*\dot{\psi} - \frac{h}{4\pi i}\dot{\psi}^*\psi - \frac{h^2}{8\pi^2 m}\nabla\psi \cdot \nabla\psi^* - V\psi^*\psi - \frac{h}{4\pi i}\psi^*\dot{\psi} + \frac{h}{4\pi i}\psi\dot{\psi}^* \\
&= -\frac{h^2}{8\pi^2 m}\nabla\psi \cdot \nabla\psi^* - V\psi^*\psi & (42)
\end{aligned}$$

4 Problem 1

The equations of motion are

$$\begin{aligned} \ddot{x} + \omega^2 x &= 0 \quad (43) \\ \ddot{y} + \alpha\omega^2 y &= 0 \quad (44) \end{aligned}$$

Part a

The energy is

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\omega^2(x^2 + \alpha y^2) \quad (45)$$

So,

$$\begin{aligned} \frac{dE}{dt} &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + m\omega^2(x\dot{x} + \alpha y\dot{y}) \\ &= m(\dot{x}(-\omega^2 x) + \dot{y}(-\alpha\omega^2 y)) + m\omega^2(x\dot{x} + \alpha y\dot{y}) \qquad \text{(using (43) and (44))} \\ &= 0 \quad (46) \end{aligned}$$

Hence the energy is conserved.

Part b

$$\begin{aligned} \frac{d\Delta}{dt} &= m(\dot{x}\ddot{x} - \dot{y}\ddot{y}) + m\omega^2(x\dot{x} - \alpha y\dot{y}) \\ &= m(\dot{x}(-\omega^2 x) - \dot{y}(-\alpha\omega^2 y)) + m\omega^2(x\dot{x} - \alpha y\dot{y}) \qquad \text{(using (43) and (44))} \\ &= 0 \quad (47) \end{aligned}$$

Hence Δ is conserved.

Part c

It can be shown that for a holonomic mechanical system, the kinetic energy is always a bilinear form of the generalized coordinates, making terms of the form $\partial L/\partial\dot{q}$ necessarily linear in the generalized velocities, whenever the potential is independent of the (generalized) velocity. In particular, for the given Lagrangian,

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x} \quad (48) \\ \frac{\partial L}{\partial \dot{y}} &= m\dot{y} \quad (49) \end{aligned}$$

Since $Q_1(x, y; \epsilon)$ and $Q_2(x, y; \epsilon)$ are point transformations, they are independent of velocities. Therefore the quantity

$$\begin{aligned} \Gamma &= \frac{\partial L}{\partial \dot{x}} \left.\frac{\partial Q_1}{\partial \epsilon}\right|_{\epsilon=0} + \frac{\partial L}{\partial \dot{y}} \left.\frac{\partial Q_2}{\partial \epsilon}\right|_{\epsilon=0} \quad (50) \\ &= m\dot{x} \left.\frac{\partial Q_1}{\partial \epsilon}\right|_{\epsilon=0} + m\dot{y} \left.\frac{\partial Q_2}{\partial \epsilon}\right|_{\epsilon=0} \quad (51) \end{aligned}$$

necessarily linear in the velocities, $\dot{x}$ and $\dot{y}$.

Part d

As justified above, any invariant quantity resulting from the symmetry of the Lagrangian under a point transformation is necessarily linear in the velocities. Since Δ is a quadratic form in the velocities, we cannot find a point transformation which leaves the Lagrangian invariant and corresponds to a Noetherian conserved current that is equal to Δ. This proves that while every invariance of a Lagrangian under a continuous point transformation yields an associated conserved quantity, the converse is not necessarily true.

For holonomic mechanical systems, the stronger statement is: *For every invariance of a Lagrangian under a continuous point transformation, there is an associated conserved quantity linear in the generalized momenta, and vice versa.*

Part e

For $\alpha = 1$, the system becomes an isotropic harmonic oscillator in 2D with the Lagrangian,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + y^2) \tag{52}$$

Due to rotational symmetry, the angular momentum

$$J_z = m(x\dot{y} - \dot{x}y) \tag{53}$$

is an invariant, which is of the form Γ. As J_z is linear in the velocities, it cannot be written as a linear combination of E and Δ (which have no linear terms in $\dot{x}$ and $\dot{y}$ at all).

5 Problem 2

Part a

The action is

$$S = \int dt\, L = \int dt\, dx\, \mathcal{L} \tag{54}$$

The Lagragian density is not explicitly dependent on the field, but only on its derivatives. So, the variation in the action is

$$\begin{aligned}
\delta S &= \delta \int dt\, dx\, \mathcal{L} && (55)\\
&= \int dt\, dx\, \delta\mathcal{L} && (56)\\
&= \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta(\partial_t \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta(\partial_x \phi) \right) && (57)\\
&= \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \qquad \text{for } \mu = 0, 1 && (58)\\
&= \int dt\, dx \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right] && (59)\\
&= -\int dt\, dx\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi && (60)
\end{aligned}$$

since the first term in (60) can be converted to a surface integral over the boundary of the (1 + 1)-spacetime region, where $\delta\phi = 0$ over the boundary. So, Hamilton's principle $\delta S = 0$ yields the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{61}$$

or

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) &= 0 & (62) \\ \Longrightarrow (\partial_t^2 - c^2 \partial_x^2)\phi(x,t) &= 0 & (63) \end{aligned}$$

Part b

From the inverse Lorentz transformations,

$$\begin{aligned} t &= \gamma\left(t' + \frac{\beta}{c}x'\right) & (64) \\ x &= \gamma\left(x' + \beta c t'\right) & (65) \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial \phi}{\partial t'} &= \frac{\partial t}{\partial t'}\frac{\partial \phi}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial \phi}{\partial x} \\ &= \gamma \frac{\partial \phi}{\partial t} + \gamma\beta c \frac{\partial \phi}{\partial x} & (66) \\ \frac{\partial \phi}{\partial x'} &= \frac{\partial t}{\partial x'}\frac{\partial \phi}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial \phi}{\partial x} \\ &= \frac{\gamma\beta}{c}\frac{\partial \phi}{\partial t} + \gamma \frac{\partial \phi}{\partial x} & (67) \end{aligned}$$

So, the Lagrangian density in the transformed frame is

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t'}\right)^2 - c^2 \left(\frac{\partial \phi}{\partial x'}\right)^2 \right] & (68) \\ &= \frac{1}{2}\left[\left(\gamma^2 \left(\frac{\partial \phi}{\partial t}\right)^2 + \gamma^2\beta^2c^2 \left(\frac{\partial \phi}{\partial x}\right)^2 + 2\gamma^2\beta c \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x} \right) \right. \\ &\quad \left. -c^2 \left(\frac{\gamma^2\beta^2}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \gamma^2 \left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{2\gamma^2\beta}{c}\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x} \right) \right] \\ &= \frac{1}{2}\left[\gamma^2(1-\beta^2)\left(\frac{\partial \phi}{\partial t}\right)^2 - \gamma^2c^2(1-\beta^2)\left(\frac{\partial \phi}{\partial x}\right)^2 \right] \\ &= \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - c^2 \left(\frac{\partial \phi}{\partial x}\right)^2 \right] \qquad \left(\text{as } \gamma = \tfrac{1}{\sqrt{1-\beta^2}}\right) & (69) \\ &= \mathcal{L} & (70) \end{aligned}$$

Hence the Lagrangian density is invariant under the Lorentz transformation.

Part c

$$
\begin{aligned}
dx\,dt &= \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial t'} \\ \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial t'} \end{vmatrix} dx'\,dt' \qquad (71)\\
&= \begin{vmatrix} \gamma & \gamma\beta c \\ \frac{\gamma\beta}{c} & \gamma \end{vmatrix} dx'\,dt' \\
&= \gamma^2(1-\beta^2)dx'\,dt' \\
&= dx'\,dt' \qquad (72)
\end{aligned}
$$

So the volume element in $(1+1)$-spacetime is Lorentz invariant. Since the Lagrangian density is also Lorentz invariant, therefore the action $S = \int dt\,dx\,\mathcal{L}$ is also a Lorentz invariant quantity.

Part d

The Euler-Lagrange equation is obtained by extremizing the action, i.e. via $\delta S = 0$. As L, the $(1+1)$-spacetime volume element as well as the Lagrangian density $\mathcal{L}$ are all Lorentz invariant quantities,

$$
\delta S = \delta \int dt\,dx\,\mathcal{L} = 0 \longleftrightarrow \delta S' = \delta \int dt'\,dx'\,\mathcal{L}' = 0 = \delta S \qquad (73)
$$

Repeating the steps carried out in part (a) with all quantities replaced by their primed counterparts, we arrive at the Euler Lagrange equation,

$$
\frac{\partial}{\partial t'}\left(\frac{\partial \mathcal{L}'}{\partial(\partial_{t'}\phi)}\right) + \frac{\partial}{\partial x'}\left(\frac{\partial \mathcal{L}'}{\partial(\partial_{x}\phi)}\right) = 0 \qquad (74)
$$

in the transformed frame. This proves that the Euler-Lagrange equations are form invariant, i.e. covariant.

In particular, using (67) and (68) we have

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial t'^2} &= \gamma^2\partial_t^2\phi + 2\gamma^2\beta c\partial_{xt}^2\phi + \gamma^2\beta^2c^2\partial_x^2 \qquad (75)\\
\frac{\partial^2\phi}{\partial x'^2} &= \frac{\gamma^2\beta^2}{c^2}\partial_t^2\phi + \frac{2\gamma^2\beta}{c}\partial_{xt}^2\phi + \gamma^2\beta^2 c\partial_x^2 \qquad (76)
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial t'^2} - c^2\frac{\partial^2\phi}{\partial x'^2} &= \gamma^2(1-\beta^2)\partial_t^2\phi - \gamma^2(1-\beta^2)\partial_x^2 \qquad (77)\\
&= \frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} \qquad (78)
\end{aligned}
$$

So, we conclude that the Euler-Lagrange equation is also invariant.

6 Problem 3

The modified action is

$$
\begin{aligned}
S' &= \int_{t_1}^{t_2} dt\, L' && (79)\\
&= \int_{t_1}^{t_2} dt \left(\frac{1}{2}m\dot{\boldsymbol{x}}^2 - q\left[\left(\phi' - \frac{d\Lambda}{dt}\right) + (\boldsymbol{A} + \nabla\Lambda)\cdot\dot{\boldsymbol{x}}\right]\right) && (80)\\
&= \int_{t_1}^{t_2} dt \left(L + q\frac{d\Lambda}{dt} - q\nabla\Lambda\cdot\dot{\boldsymbol{x}}\right) && (81)\\
&= S + q\int_{t_1}^{t_2} dt\,\frac{d\Lambda}{dt} - q\int_{t_1}^{t_2} dt\,\nabla\Lambda\cdot\dot{\boldsymbol{x}} && (82)\\
&= S + q\int_{t_1}^{t_2} dt\,\frac{d\Lambda}{dt} - q\int_{t_1}^{t_2} d\boldsymbol{x}\cdot\nabla\Lambda && (83)\\
&= S + q\int_{t=t_1}^{t_2} d\Lambda - q\int_{t=t_1}^{t_2} d\Lambda && (84)\\
&= S && (85)
\end{aligned}
$$

So, under a time dependent gauge transformation, the action is left invariant, independent of the path.