Homework 1 - Solutions†

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Goldstein 2.2

The canonical momentum p_θ is defined as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial U}{\partial \dot{\theta}} \tag{1}$$

where $T = T(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ and $U = U(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ are kinetic and potential energy of the system, which then define the Lagrangian $L = T - U$. Hence we can write

$$p_\theta = \frac{\partial}{\partial \dot{\theta}} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - \sum_i \left(\frac{\partial U}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{\theta}} + \frac{\partial U}{\partial \mathbf{v}_i} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{\theta}} \right) \tag{2}$$

If we rotate the system by angle $\delta\theta$, then the change in position vector $\mathbf{r}_i$ is $\delta \mathbf{r}_i = (\mathbf{n} \times \mathbf{r}_i)\delta\theta$. Therefore

$$\frac{\partial \mathbf{v}_i}{\partial \dot{\theta}} = \mathbf{n} \times \mathbf{r}_i, \qquad \frac{\partial \mathbf{r}_i}{\partial \dot{\theta}} = 0 \tag{3}$$

and by remembering that $\partial \mathbf{v}_i / \partial \dot{\theta} = \partial \mathbf{r}_i / \partial \theta$, we will have

$$\begin{aligned} p_\theta &= \sum_i \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{\theta}} \right) - \sum_i \frac{\partial U}{\partial \mathbf{v}_i} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{\theta}} \\ &= \sum_i \Big(m_i \mathbf{v}_i \cdot (\mathbf{n} \times \mathbf{r}_i) \Big) - \sum_i \Big(\nabla_{\mathbf{v}_i} U \cdot (\mathbf{n} \times \mathbf{r}_i) \Big) \\ &= \sum_i \mathbf{n} \cdot \mathbf{r}_i \times m_i \mathbf{v}_i - \sum_i \Big(\nabla_{\mathbf{v}_i} U \cdot (\mathbf{n} \times \mathbf{r}_i) \Big) \\ &= L_\theta - \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \nabla_{\mathbf{v}_i} U \end{aligned} \tag{4}$$

For electromagnetic potential $U = \sum_i (q_i \phi_i - \frac{q_i}{c} \mathbf{A}_i \cdot \mathbf{v}_i)$, we will get

$$p_\theta = L_\theta + \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \frac{q_i}{c} \mathbf{A}_i \tag{5}$$

Goldstein 2.4

Suppose we have two points in the sphere (θ_1, ϕ_1) and (θ_2, ϕ_2). We have to find the equation of curve which connects those two points, and prove that the curve lies on the great circle. However, it is simpler to rotate the sphere (or, redefine the coordinate system) such that those points lie on the equator of the sphere, hence they have the coordinates $(\pi/2, \phi_1)$ and $(\pi/2, \phi_2)$. Intuitively we can say that it is obvious that the curve connecting them lies on equator, and hence on the great circle. But it is necessarily important to work out the details using the variational principle.

The length of the curve is

$$\begin{aligned} S &= \int \sqrt{R^2 d\theta^2 + R^2 d\phi^2} \\ &= \int_{\phi_1}^{\phi_2} R\sqrt{1+\dot{\theta}^2} d\phi \end{aligned} \tag{6}$$

where the dot means differentiation with respect to ϕ, and R is radius of the sphere. So we have to minimize the integral of the function $\sqrt{1+\dot{\theta}^2}$, but it also would work if we minimize the integral of the function $f = 1 + \dot{\theta}^2$ along the interval $[\phi_1, \phi_2]$, although the converse is not generally true. Therefore, using the Euler-Lagrange equation,

$$\begin{aligned} \frac{d}{d\phi}\left(\frac{df}{d\dot{\theta}}\right) - \frac{\partial f}{\partial \theta} &= 0 \\ 2\ddot{\theta} - 0 &= 0 \end{aligned} \tag{7}$$

such that we have $\dot{\theta} = \text{constant} \equiv k$. It yields

$$\begin{aligned} \int_{\pi/2}^{\pi/2} d\theta &= k \int_{\phi_1}^{\phi_2} d\phi \\ 0 &= k(\phi_2 - \phi_1) \end{aligned} \tag{8}$$

and since generally we take two distinct points (i.e., $\phi_1 \neq \phi_2$), then $k = 0$, which implies $\dot{\theta} = 0$ in interval $[\phi_1, \phi_2]$. It concludes that the curve lies on the great circle.

Goldstein 2.12

The action J can be written as

$$J = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, \ddot{q}_i, t) dt \tag{9}$$

such that its variation can be calculated straightforwardly as

$$\begin{aligned}
\delta J &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i} \delta \ddot{q}_i \Big) \\
&= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) \delta q_i - \frac{d}{dt} \Big(\frac{\partial L}{\partial \ddot{q}_i} \Big) \delta \dot{q}_i \Big) dt \\
&\quad + \Big(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big) \Big|_{t_1}^{t_2} + \Big(\frac{\partial L}{\partial \ddot{q}_i} \delta \dot{q}_i \Big) \Big|_{t_1}^{t_2} \\
&= \int_{t_1}^{t^2} \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) + \frac{d^2}{dt^2} \Big(\frac{\partial L}{\partial \ddot{q}_i} \Big) \Big) \delta q_i \, dt + \Big(\frac{\partial L}{\partial \ddot{q}_i} \delta q_i \Big) \Big|_{t_1}^{t_2} \\
0 &= \int_{t_1}^{t^2} \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) + \frac{d^2}{dt^2} \Big(\frac{\partial L}{\partial \ddot{q}_i} \Big) \Big) \delta q_i \, dt
\end{aligned} \tag{10}$$

where we have omitted some terms because the variation of q_i and $\dot{q}_i$ is zero in the end points. Therefore, we have the equation of motion

$$\frac{d^2}{dt^2} \Big(\frac{\partial L}{\partial \ddot{q}_i} \Big) - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot{q}_i} \Big) + \frac{\partial L}{\partial q_i} = 0 \tag{11}$$

For the Lagrangian $L = -\frac{m}{2} q \ddot{q} - \frac{k}{2} q^2$, we first calculate

$$\frac{\partial L}{\partial \ddot{q}} = -\frac{m}{2} q, \qquad \frac{\partial L}{\partial \dot{q}} = 0, \qquad \frac{\partial L}{\partial q} = -\frac{m}{2} \ddot{q} - kq \tag{12}$$

and by plugging those equations into the equation of motion, it yields

$$\ddot{q} + \frac{k}{m} q = 0 \tag{13}$$

and of course, this is the equation of motion for the simple harmonic oscillator, for $m \neq 0$.

Additional Problem

Problem 1

a)

Before collision, the particle's velocity is $\dot{x}$. Since the wall's velocity in the lab frame is $\dot{L}$, then the velocity of particle in the wall's frame before collision is $\dot{x}-\dot{L}$. The collision is elastic, so the particle's velocity after collision in the wall's frame is $-\dot{x}+\dot{L}$. In the lab's frame, this velocity is $-\dot{x}+2\dot{L}$. Therefore, due to collision with moving wall, particle changes the velocity $\dot{x} \to -\dot{x}+2\dot{L}$.

b)

The moving wall moves slowly, in the sense that its velocity $\dot{L}$ is much less than particle's initial velocity, i.e. $\dot{L} \ll v_0$.

c)

Suppose the particle moves with velocity $\dot{x}$ instantaneously after hitting the rest wall. Then after colliding with the moving one, its velocity becomes $-\dot{x}+2\dot{L}$. After colliding with the rest wall for the second time, its velocity is $\dot{x}-2\dot{L}$. Therefore, its acceleration in this cycle of process is

$$\ddot{x} = \frac{\dot{x}-2\dot{L}-\dot{x}}{\Delta t} \approx -\frac{2\dot{L}}{2L_0/\dot{x}} \tag{14}$$

where we have used the assumption that $\dot{L} \ll v_0$, and hence $\dot{L} \ll \dot{x}$, and the assumption that $L \approx L_0$ after a time $t > 0$. Here, L_0 is the initial length of the box. Therefore,

$$\begin{aligned} \frac{d(L^2E)}{dt} &= 2L\dot{L}E + L^2\dot{E} \\ &= m\dot{x}^2 L\dot{L} + m\dot{x}\ddot{x}L^2 \\ &\approx m\dot{x}^2 L_0\dot{L} - m\dot{x}^2 L_0\dot{L} \\ &= 0 \end{aligned} \tag{15}$$

which tells us that L^2E is adiabatically invariant.

d)

Analysis using quantum mechanics principle tells us that the energy of a particle in a one-dimensional box is

$$E = \frac{n^2\hbar^2}{2mL^2}, \qquad n = 0, 1, 2, \dots \tag{16}$$

where L is length of the box. Therefore, if the wall is moving slowly, then the change of lenght, ΔL, in a particular interval of time Δt is much less than L_0. If we expand the energy expression above, then we will have

$$E = \frac{n^2\hbar^2}{2mL_0^2}\Big(1 - 2\frac{\Delta L}{L_0} + \dots\Big) \approx \frac{n^2\hbar^2}{2mL_0^2} \tag{17}$$

It explains why we get the result of adiabatically invariance of L^2E.

Problem 2

a)

Using Lagrange equation, the equation of motion for the Lagrangian $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2$ is

$$\ddot{x} + \omega_0^2 x^2 = 0 \tag{18}$$

b)

Using coordinate transformation $x = \sinh q$, the equation of motion now takes the form

$$\ddot{q} + \dot{q}^2 \tanh q + \omega_0^2 \tanh q = 0 \tag{19}$$

c)

Applying the same transformation of coordinate to Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2x^2$ will imply

$$L(q, \dot{q}) = \frac{1}{2}m\cosh^2 q\, \dot{q}^2 - \frac{1}{2}m\omega_0^2 \sinh^2 q \tag{20}$$

d)

Using the Lagrange equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} = 0 \tag{21}$$

where

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = m\cosh^2 q\,\ddot{q} + 2m\sinh q\cosh q\,\dot{q}^2 \tag{22}$$

$$\frac{\partial L}{\partial q} = m\sinh q\cosh q\,\dot{q}^2 - m\omega_0^2\sinh q\cosh q \tag{23}$$

will imply

$$\ddot{q} + \dot{q}^2\tanh q + \omega_0^2\tanh q = 0 \tag{24}$$

e)

The results of *b)* and *d)* are the same, which reflects the independency of variational principle to the specific coordinate representation.

Problem 3

a)

It is easy to verify that the family $x(t,\omega) = l\sin\omega t/\sin\omega T$ satisfies the boundary condition $x(0,\omega) = 0$ and $x(T,\omega) = l$.

b)

For $\omega = \omega_0$, the family $x(t,\omega_0) = l\sin\omega_0 t/\sin\omega_0 T$ satisfies the equation of motion $\ddot{x} + \omega_0^2 x = 0$, since technically $x(t,\omega_0) \sim \sin\omega_0 t$.

c)

We can compute the action as

$$\begin{aligned}
S(\omega) &= \int_0^T L(x,\dot{x})dt \\
&= \int_0^T \Big(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2\Big)dt \\
&= \int_0^T \left(\frac{1}{2}m\omega^2 l^2\frac{\cos^2(\omega t)}{\sin^2(\omega T)} - \frac{1}{2}m\omega_0^2 l^2\frac{\sin^2(\omega t)}{\sin^2(\omega T)}\right)dt \\
&= \frac{1}{2}\frac{ml^2}{\sin^2(\omega T)}\left((\omega^2-\omega_0^2)\frac{T}{2} + (\omega^2+\omega_0^2)\frac{\sin(2\omega T)}{4\omega}\right)
\end{aligned} \tag{25}$$

d)

Using tedious but straightforward algebra (actually I use *Mathematica* here) we can prove that

$$\frac{dS(\omega)}{d\omega}\bigg|_{\omega=\omega_0} = 0 \tag{26}$$

which reflects the vanishing of first derivative of action with respect to ω, hence the trajectory $x(t, \omega = \omega_0)$ will represent the actual trajectory.

Problem 4

a)

We can easily verify that $x(0, c) = 0$ and $x(T, c) = l$.

b)

Using the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2$, where

$$x(t, c) = l\frac{t}{T} + cl\Big(\frac{t^3}{T^3} - \frac{t}{T}\Big) \tag{27}$$

we can compute the action as

$$\begin{aligned} S(c) &= \int_0^T L(x, \dot{x})dt \\ &= \Big(\frac{2}{5}\frac{ml^2}{T} - \frac{4}{105}m\omega_0^2 l^2 T\Big)c^2 + \frac{2}{15}m\omega_0^2 l^2 Tc + \frac{1}{2}\frac{ml^2}{T} - \frac{1}{6}m\omega_0^2 l^2 T \end{aligned} \tag{28}$$

c)

Taking the differentiation of action obtained in part *b)* with respect to c, and set this first differentiation to zero, we will get

$$c = \frac{7}{2}\Big(\frac{\omega_0^2 T^2}{2\omega_0^2 T^2 - 21}\Big) \tag{29}$$

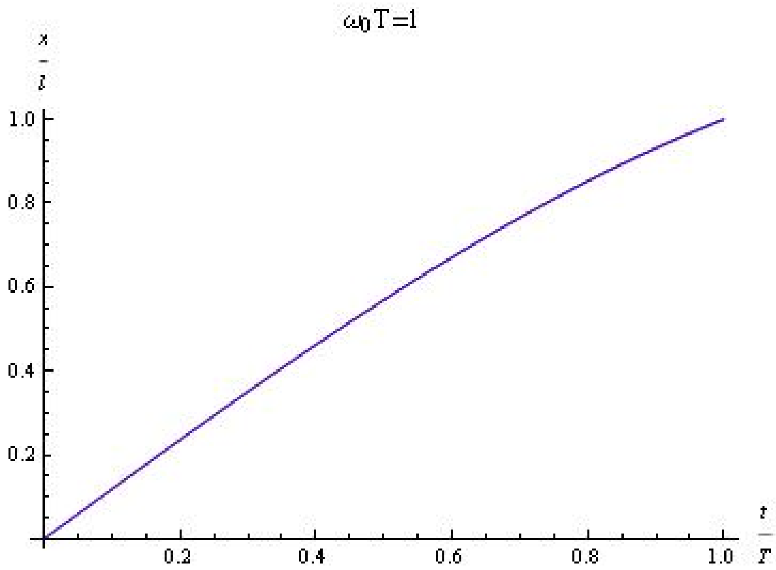
One also can prove that the second derivation of action with respect to c is positive for $\omega_0 T < \sqrt{21/2} \approx 3.2$, so our approximation is *best* if $\omega_0 T \ll 3$, with the solution takes the form

$$x(t) = l\frac{t}{T} + \frac{7}{2}\Big(\frac{\omega_0^2 T^2}{2\omega_0^2 T^2 - 21}\Big)\Big(\frac{t^3}{T^3} - \frac{t}{T}\Big)l \tag{30}$$

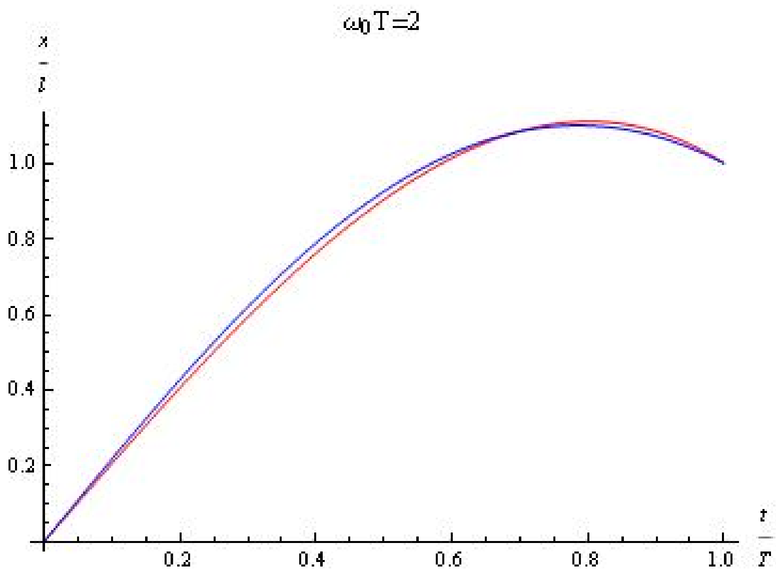
(All calculations in this section are performed in *Mathematica*)

d)

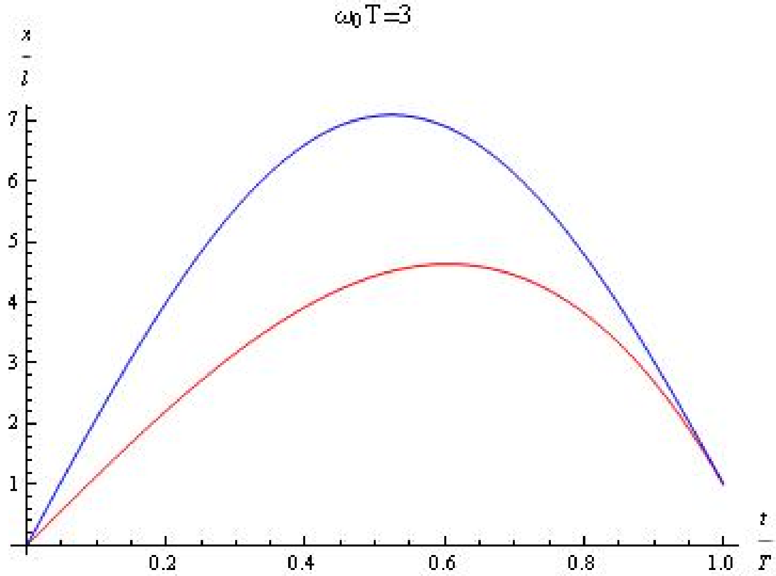
Here are graphs of each cases:



ω_0T=1
$\frac{x}{l}$
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
$\frac{t}{T}$



ω_0T=2
$\frac{x}{l}$
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
$\frac{t}{T}$



where the red plot is our function $x(t)$, and the blue one is the exact solution of the system (which is obtained from *Problem 3*). From the plot we can see that the function $x(t)$ cannot give us a good approximation if $\omega_0 T \to 3$, where the reason is, as stated in *part c)* before, the second differentiation of action $S(c)$ tends to zero when $\omega_0 T \to \sqrt{21/2}$, which makes our approximation 'not good enough' to represent the exact solution.