Theoretical Dynamics October 14, 2010

Homework 6

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Problem 1

Part (a)

The equation of motion is

$$\frac{d}{d\tau}((m+\boldsymbol{\mathcal{S}})u^\mu) = \partial^\mu \boldsymbol{\mathcal{S}} \tag{1}$$

which can be rewritten as

$$\begin{aligned}\frac{d}{d\tau}(mu^\mu) &= \partial^\mu \boldsymbol{\mathcal{S}} - \frac{d}{d\tau}(\boldsymbol{\mathcal{S}}u^\mu) \\ &= \partial^\mu \boldsymbol{\mathcal{S}} - \left\{\frac{d\boldsymbol{\mathcal{S}}}{d\tau}u^\mu + \boldsymbol{\mathcal{S}}\frac{du^\mu}{d\tau}\right\} \\ &= \partial^\mu \boldsymbol{\mathcal{S}} - \partial^\alpha \boldsymbol{\mathcal{S}} u_\alpha u^\mu - \boldsymbol{\mathcal{S}}\frac{du^\mu}{d\tau} \end{aligned} \tag{2}$$

where we have used the fact that $d\boldsymbol{\mathcal{S}}/d\tau = u^\mu g_{\mu\nu}\partial^\nu \boldsymbol{\mathcal{S}} = u_\alpha \partial^\alpha \boldsymbol{\mathcal{S}}$.

Part (b)

Multiplying both sides of the above equation by u_μ we get

$$\begin{aligned} m\frac{du^\mu}{d\tau}u_\mu &= u_\mu \partial^\mu \boldsymbol{\mathcal{S}} - \partial^\alpha \boldsymbol{\mathcal{S}} u_\alpha u^\mu u_\mu - \boldsymbol{\mathcal{S}}\frac{du^\mu}{d\tau}u_\mu \\ &= u_\mu \partial^\mu \boldsymbol{\mathcal{S}} - \partial^\alpha \boldsymbol{\mathcal{S}} u_\alpha - \boldsymbol{\mathcal{S}}\frac{du^\mu}{d\tau}u_\mu \qquad (\text{as } u^\mu u_\mu = 1) \end{aligned}$$

which implies

$$(m+\boldsymbol{\mathcal{S}})\frac{du^\mu}{d\tau}u_\mu = 0 \tag{3}$$

As $\boldsymbol{\mathcal{S}}$ is a space-time dependent scalar field, it is not identically equal to $-m$ and hence this implies

$$\frac{du^\mu}{d\tau}u_\mu = 0 \tag{4}$$

Part (c)

The action is

$$\begin{aligned} S &= \int d\tau(-m + \boldsymbol{\mathcal{S}}) & (5)\\ &= \int dt\sqrt{1-v^2}(-m+\boldsymbol{\mathcal{S}}) & (6)\\ &= \int dt\, L & (7)\end{aligned}$$

So the Lagrangian is

$$L = \sqrt{1-v^2}(-m+\boldsymbol{\mathcal{S}}) \tag{8}$$

For $\boldsymbol{\mathcal{S}} << m$ and non-relativistic conditions,

$$\begin{aligned} L &= \sqrt{1-v^2}(-m+\boldsymbol{\mathcal{S}}) \\ &\approx -\left(m - \frac{m}{2}v^2 + \boldsymbol{\mathcal{S}} + O(v^4)\right) & (9)\\ &\approx \frac{1}{2}m\dot{\vec{x}}^2 - \boldsymbol{\mathcal{S}} - m \qquad \text{where } \dot{\vec{x}} = \boldsymbol{v} & (10)\end{aligned}$$

So the Lagrangian equals $\frac{1}{2}m\dot{\vec{x}}^2 - \boldsymbol{\mathcal{S}}$ up to an irrelevant constant $-m$, which does not affect the equations of motion. The equations of motion in this regime are

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

which, from equation (10), can be written as the vector equation,

$$m\ddot{\vec{x}} = -\boldsymbol{\nabla}\boldsymbol{\mathcal{S}} \tag{11}$$

Problem 2

$$\begin{aligned} S &= \int d\tau(-m + A_\mu u^\mu) & (12)\\ &= \int dt\sqrt{1-v^2}(-m + A_\mu u^\mu) \qquad \text{as } dt = d\tau/\gamma & (13)\\ &= \int dt\, L & (14)\end{aligned}$$

where

$$L = \sqrt{1-v^2}(-m + A^\mu u_\mu) \tag{15}$$

Now,

$$\begin{aligned} L &= \sqrt{1-v^2}(-m + A^\mu u_\mu) \\ &= -m\sqrt{1-v^2} + \frac{1}{\gamma}(A^\mu g_{\mu\nu} u^\nu) \\ &= -m\sqrt{1-v^2} + \frac{1}{\gamma}(A_t\gamma - \boldsymbol{A}\cdot\gamma\boldsymbol{v}) \\ &= -m\sqrt{1-v^2} + A_t - A_j v_j & (16)\end{aligned}$$

The Euler-Lagrange equations are

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) &= \frac{\partial L}{\partial x_i} && (17)\\
\Longrightarrow \frac{d}{dt}\left(\frac{mv_i}{\sqrt{1-v^2}} - A^i\right) &= \frac{\partial A_t}{\partial x_i} - \frac{\partial A^i}{\partial x_j}v_j \\
\Longrightarrow \frac{d}{dt}\left(\frac{mv_i}{\sqrt{1-v^2}}\right) &= \frac{dA_i}{dt} + \frac{\partial A_t}{\partial x_i} - \frac{\partial A^i}{\partial x_j}v_j \\
\Longrightarrow \frac{d}{dt}(mu^i) &= \frac{\partial A^i}{\partial t} + \frac{\partial A^i}{\partial x_j}v_j + \frac{\partial A_t}{\partial x_i} - \frac{\partial A_j}{\partial x_i}v_j \\
\Longrightarrow \frac{d}{d\tau}(mu^i) &= \frac{\partial A^i}{\partial t}\gamma + \frac{\partial A^i}{\partial x_j}\gamma v_j + \frac{\partial A_t}{\partial x_i}\gamma - \frac{\partial A_j}{\partial x_i}\gamma v_j \\
\Longrightarrow \frac{d}{d\tau}(mu^i) &= \left(\frac{\partial A^i}{\partial t}\gamma + \frac{\partial A^i}{\partial x_j}\gamma v_j\right) + \left(\frac{\partial A_t}{\partial x_i}\gamma - \frac{\partial A_j}{\partial x_i}\gamma v_j\right) \\
\Longrightarrow \frac{d}{d\tau}(mu^i) &= \frac{\partial A^i}{\partial x_\nu}u_\nu - \frac{\partial A^\nu}{\partial x_i}u_\nu && (18)\\
\Longrightarrow \frac{d}{d\tau}(mu^i) &= (\partial^\nu A^i - \partial^i A^\nu)u_\nu && (19)
\end{aligned}
$$

So the equation of motion is

$$
\frac{d}{d\tau}(mu_\mu) = \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right)u^\nu \qquad (20)
$$

Problem 3

Part (a)

For A^μ to transform as a 4-vector, we must have

$$
A^\mu \longrightarrow A'^\mu = L^\mu{}_\nu A^\nu \qquad (21)
$$

where $L^\mu{}_\nu$ is the Lorentz transformation. Now, under a gauge transformation,

$$
A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu G \qquad (22)
$$

So, under a gauge transformation, the field tensor $F^{\mu\nu}$ is invariant, and hence the fields (which are derivable from the field tensor) are also invariant:

$$
\begin{aligned}
F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu && (23)\\
&= \partial^\mu(A^\nu + \partial^\nu G) - \partial^\nu(A^\mu + \partial^\mu G) \\
&= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\nu\partial^\mu G - \partial^\nu\partial^\mu G \\
&= \partial^\mu A^\nu - \partial^\nu A^\mu \\
&= F^{\mu\nu} && (24)
\end{aligned}
$$

So, the invariance of the field tensor under a gauge transformation holds for arbitrary scalar functions G which are (at least) twice differentiable in the space-time coordinates. Now, for a gauge transformation to be similar to a Lorentz transformation for some G, we examine the condition

$$L^\mu{}_\nu A^\nu \stackrel{?}{=} A^\mu + \partial^\mu G \tag{25}$$

which implies

$$(L^\mu{}_\nu - \delta^\mu{}_\nu) A^\nu \stackrel{?}{=} \partial^\mu G \tag{26}$$

This implies that *if* G is chosen so as to satisfy this condition, then A^μ will transform as a four vector under both the Lorentz transformation as well as the gauge transformation (but for this particular G only). Clearly, since the choice of G is arbitrary in a gauge transformation (and not related to A^μ), any other choice of G which does not satisfy equation (26) will be a valid gauge transformation, but will not result in A^μ transforming like a 4-vector. In other words, since A^μ can always be gauge-transformed by arbitrary choices of scalar functions G as in equation (22), it does not necessarily transform as a 4-vector under Lorentz transformations.

Part (b)

The right hand side of the equation $\partial_\mu A'^\mu = 0$ is a scalar, so it is also Lorentz invariant. So, the left hand side must be Lorentz invariant as well. This is only possible if the left hand side transforms like the scalar product of two 4-vectors. We know that ∂_μ is a four-vector operator, that is, it is an operator which transforms like a covariant 4-vector under a Lorentz transformation. Therefore, A'^μ must transform like a 4-vector. (The regularizing conditions $|A^\mu| \to 0$ as $|x| \to \infty$ ensure that the fields are well behaved and the scalar product is well defined, under the gauge transformation.)

Part (c)

The condition $\partial_\mu A'^\mu = 0$ implies $\partial_\mu A^\mu = -\partial_\mu \partial^\mu G$. Given a vector potential A^μ in an arbitrary gauge, it is always possible to solve this second order differential equation to find a scalar function G. Then, the particular gauge choice $A'^\mu = A^\mu + \partial^\mu G$ corresponds to A'^μ transforming like a 4-vector.

Part (d)

Uunder a gauge transformation, the field tensor $F^{\mu\nu}$ is invariant:

$$\begin{aligned}
F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu && (27)\\
&= \partial^\mu (A^\nu + \partial^\nu G) - \partial^\nu (A^\mu + \partial^\mu G)\\
&= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\nu \partial^\mu G - \partial^\nu \partial^\mu G\\
&= \partial^\mu A^\nu - \partial^\nu A^\mu\\
&= F^{\mu\nu} && (28)
\end{aligned}$$

Part (e)

The field tensor is

$$F^{\mu\nu} \quad = \quad \partial^\mu A^\nu - \partial^\nu A^\mu \tag{29}$$

Under a Lorentz transformation,

$$A^\mu \longrightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu \tag{30}$$

$$\partial^\mu \longrightarrow \partial'^\mu = \Lambda^\mu{}_\nu \partial^\nu \tag{31}$$

So the Lorentz transformed field tensor is

$$\begin{aligned}
F'^{\mu\nu} &= \partial'^\mu A'^\nu - \partial'^\nu A'^\mu && (32)\\
&= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial^\alpha A^\beta - \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial^\beta A^\alpha && (33)\\
&= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) && (34)\\
&= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} && (35)
\end{aligned}$$

Part (f)

This is most easily seen by considering the action,

$$S \quad = \quad \int d\tau(-m + qA^\mu u_\mu) \tag{36}$$

This is the action integral that we started out with in problem 2 with the charge q factored in. Using the result of problem 2, we get,

$$\begin{aligned}
\frac{d}{d\tau}(mu^\mu) &= q\left(\frac{\partial A^\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right)u^\nu \\
&= q\left(\partial_\nu A^\mu - \partial^\mu A_\nu\right)u^\nu \\
&= q\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right)u_\nu && (37)\\
&= qF^{\mu\nu}u_\nu && (38)
\end{aligned}$$

which is the required equation of motion.

Problem 4

Contracting both sides of the equation of motion with u_μ we get

$$\begin{aligned}
m\frac{du^\mu}{d\tau}u_\mu &= qF^{\mu\nu}u_\nu u_\mu && (39)\\
&= qF^{\nu\mu}u_\nu u_\mu \\
&= -qF^{\mu\nu}u_\nu u_\mu \\
&= 0 && (40)
\end{aligned}$$

due to the antisymmetry of the field tensor $F^{\mu\nu}$. Therefore, since $m \neq 0$, we get $\frac{du^\mu}{d\tau}u_\mu = 0$.

Problem 5

Part (a)

$$\begin{aligned}\partial_0 A^0 &= \frac{\partial \Phi}{\partial t} = 0 &(41)\\ \partial_1 A^1 &= \frac{\partial(-E_0 t)}{\partial x} = 0 &(42)\\ \partial_2 A^2 &= \frac{\partial(0)}{\partial y} = 0 &(43)\\ \partial_3 A^3 &= \frac{\partial(0)}{\partial z} = 0 &(44)\end{aligned}$$

So, $\partial_\mu A^\mu = 0$.

Part (b)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{45}$$

The field tensor has only two non-vanishing terms $F^{01} = -F^{10} = -E_0$. So, the field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{46}$$

The equation of motion is

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu \tag{47}$$

Using the result of problem 4(d) of homework 5 (with $g \to qE_0/m$), the position is

$$x(t) = \frac{m}{qE_0}\left[\sqrt{1 + \left(\frac{q^2 E_0^2}{m^2}\right) t^2} - 1\right] \tag{48}$$

Problem 6

Part (a)

The Green's function in time domain can be written as

$$G(t,t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-t')}}{(\omega - i\beta)^2 + (\beta^2 - \omega_0^2)} \tag{49}$$

So the poles are at

$$\begin{aligned}\omega_1 &= -\sqrt{\omega_0^2 - \beta^2} + i\beta &(50)\\ \omega_2 &= \sqrt{\omega_0^2 - \beta^2} + i\beta &(51)\end{aligned}$$

Assuming $0 < \beta < \omega_0$, both poles lie in the upper half-plane.

Case I: $t > t'$

In this case, the integral is evaluated in the upper half plane, picking up contributions from both poles. This is because on the semicircular contour, $|e^{i\omega(t-t')}| = e^{-Im(\omega)(t-t')}$ is bounded if $Im(\omega) > 0$ (where Im denotes the imaginary part). Applying the residue theorem, we get

$$\begin{aligned} G(t,t') &= -\frac{1}{2\pi} \times 2\pi i \left[\underset{\omega=\omega_1}{\text{Res}} \frac{e^{i\omega(t-t')}}{(\omega-\omega_1)(\omega-\omega_2)} + \underset{\omega=\omega_2}{\text{Res}} \frac{e^{i\omega(t-t')}}{(\omega-\omega_1)(\omega-\omega_2)} \right] && (52) \\ &= \frac{i}{\omega_1 - \omega_2} \left(e^{i\omega_2(t-t')} - e^{i\omega_1(t-t')} \right) \\ &= -\frac{i}{2\sqrt{\omega_0^2 - \beta^2}} e^{-\beta(t-t')} \left(e^{i\sqrt{\omega_0^2-\beta^2}(t-t')} - e^{-i\sqrt{\omega_0^2-\beta^2}(t-t')} \right) && (53) \\ &= \frac{1}{\sqrt{\omega_0^2-\beta^2}} e^{-\beta(t-t')} \sin\left(\sqrt{\omega_0^2-\beta^2}(t-t') \right) && (54) \end{aligned}$$

Case II: $t < t'$

In this case, on the semicircular contour, $|e^{i\omega(t-t')}| = e^{-Im(\omega)(t-t')}$ is bounded only if $Im(\omega) < 0$. So, the contour of integration must lie in the lower half plane. But the lower half plane does not contain any poles of the integrand. So, the integral is zero for $t < t'$. This is consistent with causality: the impulse input to the system comes at $t = t'$ and is zero for $t < t'$, so the response to this input cannot arrive before the input itself.

Combining the results, the Green's function can be written as

$$G(t,t') = \frac{1}{\sqrt{\omega_0^2-\beta^2}} e^{-\beta(t-t')} \sin\left(\sqrt{\omega_0^2-\beta^2}(t-t') \right) \theta(t-t') \qquad (55)$$

where $\theta(t-t')$ denotes the Heaviside step function, defined by

$$\theta(t-t') = \begin{cases} 1 & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases} \qquad (56)$$

Part (b)

For the given forcing function, the particular solution is

$$
\begin{aligned}
x_p(t) &= \int_{-\infty}^{\infty} dt'\, G(t,t')f(t') &(57)\\
&= \int_{-\infty}^{\infty} dt' \frac{1}{\omega_1-\omega_2}\left(e^{i\omega_2(t-t')}-e^{i\omega_1(t-t')}\right)\theta(t-t')f_0 e^{-\Gamma t'}\theta(t') &(58)\\
&= \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\int_0^t dt'\,\left\{e^{i\omega_2(t-t')-\Gamma t'}-e^{i\omega_1(t-t')-\Gamma t'}\right\} &(59)\\
&= \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\left(e^{i\omega_2 t}\int_0^t dt'\,e^{-(\Gamma+i\omega_2)t'}-e^{i\omega_1 t}\int_0^t dt'\,e^{-(\Gamma+i\omega_1)t'}\right) &(60)\\
&= \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\left(e^{i\omega_2 t}\frac{1-e^{-(\Gamma+i\omega_2)t}}{\Gamma+i\omega_2}-e^{i\omega_1 t}\frac{1-e^{-(\Gamma+i\omega_1)t}}{\Gamma+i\omega_1}\right) &(61)\\
&= \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\left(\frac{e^{i\omega_2 t}}{\Gamma+i\omega_2}-\frac{e^{i\omega_1 t}}{\Gamma+i\omega_1}+\frac{i(\omega_2-\omega_1)}{(\Gamma+i\omega_1)(\Gamma+i\omega_2)}e^{-\Gamma t}\right) &(62)
\end{aligned}
$$

The solution to the homogeneous equation is of the form

$$
x_h(t) = C_1 e^{i\omega_1 t}+C_2 e^{i\omega_2 t} = C_1 e^{-i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}+C_2 e^{i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t} \qquad (63)
$$

where C_1 and C_2 are constants to be fixed by boundary conditions. Therefore, the general solution can be written as

$$
\begin{aligned}
x(t) &= C_1 e^{-i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}+C_2 e^{i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}\\
&+ \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\left(\frac{e^{i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}}{\Gamma+i\omega_2}-\frac{e^{-i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}}{\Gamma+i\omega_1}+\frac{i(\omega_2-\omega_1)}{(\Gamma+i\omega_1)(\Gamma+i\omega_2)}e^{-\Gamma t}\right) \quad (64)
\end{aligned}
$$

Note: The particular solution can also be obtained by noting that it equals the convolution of the retarded Green's function for the oscillator with the forcing function, and so, the Fourier transform of the particular solution equals the product of the (phase shifted) frequency domain Green's function with the Fourier transform of the forcing function. The retarded Green's function can be obtained by Fourier transforming the differential equation of motion, with the forcing function equal to the impulse.

Part (c)

For $\Gamma >> \omega_0 >> \beta$,

$$
\begin{aligned}
\Gamma+i\omega_2 &= \Gamma+i(\sqrt{\omega_0^2-\beta^2}+i\beta)\\
&= (\Gamma-\beta)+i\sqrt{\omega_0^2-\beta^2}\\
&\approx \Gamma+i\omega_0 &(65)\\
\Gamma+i\omega_1 &= \Gamma+i(-\sqrt{\omega_0^2-\beta^2}+i\beta)\\
&= (\Gamma-\beta)-i\sqrt{\omega_0^2-\beta^2}\\
&\approx \Gamma-i\omega_0 &(66)
\end{aligned}
$$

So, in this regime, the particular solution is

$$
\begin{aligned}
x_p(t) &= \frac{if_0\,\theta(t)}{\omega_1-\omega_2}\left(\frac{e^{i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}}{\Gamma+i\omega_2}-\frac{e^{-i\sqrt{\omega_0^2-\beta^2}t}e^{-\beta t}}{\Gamma+i\omega_1}+\frac{i(\omega_2-\omega_1)}{(\Gamma+i\omega_1)(\Gamma+i\omega_2)}e^{-\Gamma t}\right)\\
&\approx \frac{if_0\,\theta(t)}{-2\omega_0}\left(e^{-\beta t}\left[\frac{e^{-i\omega_0 t}}{\Gamma+i\omega_0}-\frac{e^{i\omega_0 t}}{\Gamma-i\omega_0}\right]+\frac{2i\omega_0}{\Gamma^2+\omega_0^2}e^{-\Gamma t}\right)\\
&\approx \frac{if_0\,\theta(t)}{-2\omega_0}\left(-2ie^{-\beta t}\frac{\omega_0\cos(\omega_0 t)+\Gamma\sin(\omega_0 t)}{\Gamma^2+\omega_0^2}+\frac{2i\omega_0}{\Gamma^2+\omega_0^2}e^{-\Gamma t}\right)\\
&\approx \frac{if_0\,\theta(t)}{2\omega_0}\left(2ie^{-\beta t}\left(\frac{\omega_0\cos(\omega_0 t)+\Gamma\sin(\omega_0 t)}{\Gamma^2+\omega_0^2}\right)-\frac{2i\omega_0}{\Gamma^2+\omega_0^2}e^{-\Gamma t}\right)
\end{aligned}
\tag{67}
$$

whereas the solution to the homogeneous equation is

$$x_h(t) \approx \left(C_1e^{-i\omega_0 t}+C_2e^{i\omega_0 t}\right)e^{-\beta t} \tag{68}$$

Therefore, the general solution is

$$x(t) = \left(C_1e^{-i\omega_0 t}+C_2e^{i\omega_0 t}\right)e^{-\beta t}+\frac{if_0\,\theta(t)}{2\omega_0}\left(2ie^{-\beta t}\left(\frac{\omega_0\cos(\omega_0 t)+\Gamma\sin(\omega_0 t)}{\Gamma^2+\omega_0^2}\right)-\frac{2i\omega_0}{\Gamma^2+\omega_0^2}e^{-\Gamma t}\right) \tag{69}$$

As $\Gamma >> \omega_0 >> \beta$, the third term in the general solution, which contains $e^{-\Gamma t}$, constitutes the transient part which decays much faster than the other two terms which involve $e^{-\beta t}$. So after about 3-4 time constants of Γ, i.e. in about $3/\Gamma$ to $4/\Gamma$ time units, the general solution is of the form

$$x(t) \approx \left(C_1e^{-i\omega_0 t}+C_2e^{i\omega_0 t}\right)e^{-\beta t}+\frac{if_0\,\theta(t)}{2\omega_0}\left[2ie^{-\beta t}\left(\frac{\omega_0\cos(\omega_0 t)+\Gamma\sin(\omega_0 t)}{\Gamma^2+\omega_0^2}\right)\right] \tag{70}$$

This bears the form of the general solution of an underdamped ($\beta << \omega_0$) harmonic oscillator with the forcing function being an impulse at $t = 0$. This is also evident if we compare the particular solution in the regime $\Gamma >> \omega_0 >> \beta$ with the Green's function $G(t, t')$ obtained in part (a), and observe that the expressions are similar in this regime, if $t' = 0$. The arrival of the input impulse at $t = 0$ is signatured by the appearance of $\theta(t)$ in the second term of the general solution above.