Theoretical Dynamics October 01, 2010

Homework 4

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Goldstein 9.7

Part (a)

$F_1(q,Q,t) \longrightarrow F_2(q,P,t)$

$$-P_i = \frac{\partial F_1}{\partial Q_i} \quad (1)$$
$$F_2(q,P,t) = F_1(q,Q,t) + P_i Q_i \quad (2)$$

$F_1(q,Q,t) \longrightarrow F_3(p,Q,t)$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad (3)$$
$$F_3(p,Q,t) = F_1(q,Q,t) - p_i q_i \quad (4)$$

$F_1(q,Q,t) \longrightarrow F_4(p,P,t)$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad (5)$$
$$P_i = -\frac{\partial F_1}{\partial Q_i} \quad (6)$$
$$F_4(p,P,t) = F_1(q,Q,t) - p_i q_i + P_i Q_i \quad (7)$$

$F_2(q,P,t) \longrightarrow F_3(p,Q,t)$

$$p_i = \frac{\partial F_2}{\partial q_i} \quad (8)$$
$$Q_i = \frac{\partial F_2}{\partial P_i} \quad (9)$$
$$F_3(p,Q,t) = F_2(q,P,t) - p_i q_i - P_i Q_i \quad (10)$$

$F_2(q,P,t) \longrightarrow F_4(p,P,t)$

$$p_i = \frac{\partial F_2}{\partial q_i} \quad (11)$$
$$F_4(p,P,t) = F_2(q,P,t) - p_i q_i \quad (12)$$

$F_3(p, Q, t) \longrightarrow F_4(p, P, t)$

$$\begin{aligned} P_i &= -\frac{\partial F_3}{\partial Q_i} && (13) \\ F_4(p, P, t) &= F_3(p, Q, t) + P_i Q_i && (14) \end{aligned}$$

Part (b)

For an identity transformation, $F_2 = q_i P_i$ and by equation (7), the type 4 generating function is

$$\begin{aligned} F_4(p, P, t) &= F_2(q, P, t) - p_i q_i && (15) \\ &= q_i P_i - p_i q_i \\ &= 0 \qquad \text{as } p_i = \frac{\partial F_2}{\partial q_i} = P_i \end{aligned}$$

For an exchange transformation, $F_1 = q_i Q_i$ and by equation (4), the type 3 generating function is

$$\begin{aligned} F_3(p, Q, t) &= F_1(q, Q, t) - p_i q_i && (16) \\ &= q_i Q_i - p_i q_i \\ &= 0 \qquad \text{as } p_i = \frac{\partial F_1}{\partial q_i} = Q_i && (17) \end{aligned}$$

Part (c)

Consider a type 2 generating function $F_2(q, P, t)$ of the old coordinates and the new momenta, of the form

$$F_2(q, P, t) = f_i(q_1, \dots, q_n; t) P_i - g(q_1, \dots, q_n; t) \tag{18}$$

where f_i's are a set of independent functions, and g_i's are differentiable functions of the old coordinates and time. The new coordinates Q_i are given by

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \dots, q_n; t) \tag{19}$$

In particular, the function

$$f_i(q_1, \dots, q_n; t) = R_{ij} q_j \tag{20}$$

where R_{ij} is the (i, j)-th element of a $N \times N$ orthogonal matrix, generates an orthogonal transformation of the coordinates. Now,

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j} P_i - \frac{\partial g}{\partial q_j} = R_{ij} P_i - \frac{\partial g}{\partial q_j} \tag{21}$$

This equation can be written in matrix form, as

$$\boldsymbol{p} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} \boldsymbol{P} - \frac{\partial g}{\partial \boldsymbol{q}} \tag{22}$$

where $\boldsymbol{p}$ denotes the $N \times 1$ column vector $(p_1, \ldots, p_N)^T$, $\partial g/\partial \boldsymbol{q}$ denotes the $N \times 1$ column vector $(\partial g/\partial q_1, \ldots, \partial g/\partial q_n)^T$, and $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}$ denotes the $N \times N$ matrix with entries

$$\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right)_{ij} = \frac{\partial f_i}{\partial q_j} = R_{ij} \tag{23}$$

From (22), the new momenta are given by

$$\begin{aligned}
\boldsymbol{P} &= \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right)^{-1} \left(\boldsymbol{p} + \frac{\partial g}{\partial \boldsymbol{q}}\right) && (24)\\
&= \boldsymbol{R}^{-1} \left(\boldsymbol{p} + \frac{\partial g}{\partial \boldsymbol{q}}\right) && (25)\\
&= \boldsymbol{R}^{-1} \left(\boldsymbol{p} + \boldsymbol{\nabla}_{\boldsymbol{q}} g\right) && (26)
\end{aligned}$$

As $\boldsymbol{R}$ is an orthogonal matrix, $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, so $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$ is also an orthogonal transformation.

This gives the required result: the new momenta are given by the orthogonal transformation (R^{-1}) of an n-dimensional vector $(\boldsymbol{p} + \boldsymbol{\nabla}_{\boldsymbol{q}} g)$, whose components are the old momenta $(\boldsymbol{p})$ plus a gradient in configuration space $(\boldsymbol{\nabla}_{\boldsymbol{q}} g)$.

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Part (a)

The given Hamiltonian is

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{27}$$

The equation of motion for q is

$$\dot{q} = \frac{\partial H}{\partial p} = pq^4 \tag{28}$$

Part (b)

Suppose we let $Q^2 = 1/q^2$ and $P^2 = p^2 q^4$. Then, $Q = \pm 1/q$ and $P = \pm pq^2$. Now,

$$\begin{aligned}
\{Q, P\} &= \{\pm 1/q, \pm pq^2\}\\
&= \{q^{-1}, pq^2\}\\
&= \{q^{-1}, p\}q^2 + p\{q^{-1}, q^2\}\\
&= \left(\frac{\partial q^{-1}}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial q^{-1}}{\partial p}\frac{\partial p}{\partial q}\right) q^2 + p \times 0\\
&= \left(-\frac{1}{q^2}\right) q^2\\
&= -1
\end{aligned}$$

So, the signs on both Q and P cannot be identical. We take

$$\begin{aligned} Q &= -\frac{1}{q} \quad (29) \\ P &= pq^2 \quad (30) \end{aligned}$$

which is a valid canonical transformation. This gives the Hamiltonian,

$$H(Q,P) = \frac{1}{2}(P^2 + Q^2) \quad (31)$$

The equations of motion are

$$\begin{aligned} \dot{Q} &= \frac{\partial H}{\partial P} = P \quad (32) \\ \dot{P} &= -\frac{\partial H}{\partial Q} = -Q \quad (33) \end{aligned}$$

So, $\ddot{Q} + Q = 0$, the solution to which is of the form $Q = A\cos t + B\sin t$. This gives $P = \dot{Q} = B\cos t - A\sin t$. Now,

$$\begin{aligned} q &= -\frac{1}{Q} = -(A\cos t + B\sin t)^{-1} \quad (34) \\ p &= PQ^2 = (B\cos t - A\sin t)(A\cos t + B\sin t)^2 \quad (35) \end{aligned}$$

so,

$$\dot{q} = (A\cos t + B\sin t)^{-2}(-A\sin t + B\cos t) \quad (36)$$

and hence

$$pq^4 = (B\cos t - A\sin t)(A\cos t + B\sin t)^2(A\cos t + B\sin t)^{-4} = (B\cos t - A\sin t)(A\cos t + B\sin t)^{-2} = \dot{q} \quad (37)$$

So, the solution to the transformed equation for Q satisfies the original equation of motion for q.

Problem 1

Part (a)

$$\begin{aligned} \{X, P_x\} &= \{x + \epsilon, p_x\} \\ &= \{x, p_x\} \\ &= 1 \quad (38) \end{aligned}$$

$$\begin{aligned} \{Y, P_y\} &= \{y, p_x\} \\ &= 1 \quad (39) \end{aligned}$$

$$\begin{aligned}\{Z, P_z\} &= \{z, p_z\} \\ &= 1\end{aligned} \tag{40}$$

$$\{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \tag{41}$$

$$\{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_z\} = \{P_z, P_z\} = \{P_x, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \tag{42}$$

So, this is a canonical transformation. It corresponds to a translated canonical coordinate system (translation along the x-direction in phase space).

$$\frac{dX}{d\epsilon} = [X, P_x] = 1 \tag{43}$$

So P_x is the generator of the canonical transformation.

Part (b)

$$\begin{aligned}\{X, P_x\} &= \{x\cos\epsilon + y\sin\epsilon, p_x\cos\epsilon + p_y\sin\epsilon\} \\ &= \cos^2\epsilon\{x, p_x\} + \sin^2\{y, p_y\} \\ &= 1\end{aligned} \tag{44}$$

$$\begin{aligned}\{Y, P_y\} &= \{-x\sin\epsilon + y\cos\epsilon, -p_x\sin\epsilon + p_y\cos\epsilon\} \\ &= \sin^2\epsilon\{x, p_x\} + \cos^2\epsilon\{y, p_y\} \\ &= 1\end{aligned} \tag{45}$$

$$\begin{aligned}\{Z, P_z\} &= \{z, p_z\} \\ &= 1\end{aligned} \tag{46}$$

Using properties of the Poisson Bracket, we also have

$$\{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \tag{47}$$

$$\{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_z\} = \{P_z, P_z\} = \{P_x, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \tag{48}$$

So, this is a canonical transformation. It corresponds to a rotation about the z-axis in phase space.

$$\frac{dX}{d\epsilon} = -x\sin\epsilon + y\cos\epsilon \tag{49}$$

whereas

$$\{X, L_z\} = \{x\cos\epsilon + y\sin\epsilon, xp_y - yp_x\} = x\sin\epsilon - y\cos\epsilon \tag{50}$$

So,

$$\frac{dX}{d\epsilon} = \{X, -L_z\} \tag{51}$$

Therefore, $-L_z$ is the generator of the canonical transformation.

Part (c)

$$\begin{aligned} \{X, P_x\} &= \{x, p_x + \epsilon\} \\ &= 1 \end{aligned} \tag{52}$$

$$\begin{aligned} \{Y, P_y\} &= \{y, p_y\} \\ &= 1 \end{aligned} \tag{53}$$

$$\begin{aligned} \{Z, P_z\} &= \{z, p_z\} \\ &= 1 \end{aligned} \tag{54}$$

Using properties of the Poisson Bracket, we also have

$$\{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \tag{55}$$

$$\{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_z\} = \{P_z, P_z\} = \{P_x, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \tag{56}$$

So, this is a canonical transformation. It corresponds to a translation along the p_x direction in phase space. Now,

$$\{P_x, -X\} = -\left(\frac{\partial P_x}{\partial q_i}\frac{\partial X}{\partial p_i} - \frac{\partial P_x}{\partial p_i}\frac{\partial X}{\partial q_i}\right) = 1 = \frac{dP_x}{d\epsilon} \tag{57}$$

Therefore, $-X$ is the generator of the canonical transformation.

Part (d)

$$\begin{aligned} \{X, P_x\} &= \{(1+\epsilon)x, (1+\epsilon)^{-1}p_x\} \\ &= 1 \end{aligned} \tag{58}$$

$$\begin{aligned} \{Y, P_y\} &= \{(1+\epsilon)y, (1+\epsilon)^{-1}p_y\} \\ &= 1 \end{aligned} \tag{59}$$

$$\begin{aligned} \{Z, P_z\} &= \{(1+\epsilon)z, (1+\epsilon)^{-1}p_z\} \\ &= 1 \end{aligned} \tag{60}$$

Using properties of the Poisson Bracket, we also have

$$\{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \tag{61}$$

$$\{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_z\} = \{P_z, P_z\} = \{P_x, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \tag{62}$$

So, this is a canonical transformation. It is a scaling transformation, which preserves the volume element in phase space. Suppose g is the generator of the scaling transformation. Then,

$$\frac{\partial X}{\partial \epsilon} = x = [X, g] = [(1+\epsilon)x, g] \tag{63}$$

which implies

$$\frac{x}{1+\epsilon} = [x, g] = \frac{\partial x}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial g}{\partial x} \tag{64}$$

that is,

$$\frac{x}{1+\epsilon} = \frac{\partial g}{\partial p_x} \tag{65}$$

the solution to which is

$$g = \frac{x p_x}{1+\epsilon} + f(y, z, p_y, p_z) \tag{66}$$

As $dY/d\epsilon = y = [(1+\epsilon)y, g]$ and $dZ/d\epsilon = z = [(1+\epsilon)z, g]$, following a similar argument for Y and Z (or by symmetry) we get

$$g = \frac{x p_x}{1+\epsilon} + \frac{y p_y}{1+\epsilon} + \frac{z p_z}{1+\epsilon} + \text{ constant} \tag{67}$$

as the generator of the scaling transformation.

Problem 2

As $\boldsymbol{\eta}$ is a canonical transformation, we have

$$\frac{\partial \eta_i}{\partial \epsilon} = \{\eta_i, g\} \tag{68}$$

So,

$$\begin{aligned}
\frac{\partial H}{\partial \epsilon} &= \frac{\partial H}{\partial \eta_i}\frac{\partial \eta_i}{\partial \epsilon} \\
&= \frac{\partial H}{\partial \eta_i}\{\eta_i, g\} \\
&= \frac{\partial H}{\partial \eta_i}\frac{\partial \eta_i}{\partial \xi_j} J_{jk} \frac{\partial g}{\partial \xi_k} \qquad \text{(as } \boldsymbol{\xi} \text{ is a canonical transformation)} \\
&= \frac{\partial H}{\partial \eta_i}\frac{\partial \eta_i}{\partial \eta_j} J_{jk} \frac{\partial g}{\partial \eta_k} \qquad \text{(as Poisson Brackets are invariant under canonical transformations)} \\
&= \frac{\partial H}{\partial \eta_i}\delta_{ij} J_{jk} \frac{\partial g}{\partial \eta_k} \\
&= \frac{\partial H}{\partial \eta_i} J_{ij} \frac{\partial g}{\partial \eta_j} \\
&= \{H, g\} \\
&= -\dot{g}
\end{aligned} \tag{69}$$

But since H is conserved, $\frac{\partial H}{\partial \epsilon} = 0$ and hence $\dot{g} = 0$. Therefore, g is conserved.

Problem 3

The quantity Δ, which was found to be an invariant of the system, can be expressed in terms of the canonical coordinates x, y, p_x, p_y as

$$\Delta(x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \tag{70}$$

As Δ is the conserved generator of a family of canonical transformations parametrized by an infinitesimal parameter ϵ, we must have

$$\begin{aligned} \delta x &= \epsilon\{x, \Delta\} && (71) \\ \delta y &= \epsilon\{y, \Delta\} && (72) \\ \delta p_x &= \epsilon\{p_x, \Delta\} && (73) \\ \delta p_y &= \epsilon\{p_x, \Delta\} && (74) \\ & && (75) \end{aligned}$$

We consider each condition separately below.

$$\begin{aligned} \delta x &= \epsilon\{x, \Delta\} \\ &= \epsilon\{x, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2)\} \\ &= \epsilon\{x, \frac{p_x^2}{2m}\} \\ &= \frac{\epsilon p_x}{m} && (76) \end{aligned}$$

$$\begin{aligned} \delta y &= \epsilon\{y, \Delta\} \\ &= \epsilon\{y, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2)\} \\ &= \epsilon\{y, -\frac{p_y^2}{2m}\} \\ &= -\frac{\epsilon p_y}{m} && (77) \end{aligned}$$

$$\begin{aligned} \delta p_x &= \epsilon\{p_x, \Delta\} \\ &= \epsilon\{p_x, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2)\} \\ &= \epsilon\{p_x, \frac{1}{2}m\omega^2 x^2\} \\ &= -\epsilon m\omega^2 x && (78) \end{aligned}$$

$$\begin{aligned} \delta p_y &= \epsilon\{p_y, \Delta\} \\ &= \epsilon\{p_y, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2)\} \\ &= \epsilon\{p_y, -\frac{1}{2}m\omega^2\alpha y^2\} \\ &= \epsilon m\omega^2\alpha y && (79) \end{aligned}$$

Now, let $\epsilon = \delta\theta$ where θ is a parameter. Then, the above equations become

$$\frac{dx}{d\theta} = \frac{p_x}{m} \tag{80}$$

$$\frac{dy}{d\theta} = -\frac{p_y}{m} \tag{81}$$

$$\frac{dp_x}{d\theta} = -m\omega^2 x \tag{82}$$

$$\frac{dp_y}{d\theta} = m\omega^2 \alpha y \tag{83}$$

So,

$$\frac{d^2x}{d\theta^2} + \omega^2 x = 0 \tag{84}$$

$$\frac{d^2y}{d\theta^2} + \omega^2 \alpha y = 0 \tag{85}$$

The solutions to which are

$$x = A\cos(\omega\theta) + B\sin(\omega\theta) \tag{86}$$

$$y = C\cos(\omega\sqrt{\alpha}\theta) + D\sin(\omega\sqrt{\alpha}\theta) \tag{87}$$

and correspondingly

$$p_x = -m\omega A\sin(\omega\theta) + m\omega B\cos(\omega\theta) \tag{88}$$

$$p_y = m\omega\sqrt{\alpha}C\sin(\omega\sqrt{\alpha}\theta) - m\omega\sqrt{\alpha}D\cos(\omega\sqrt{\alpha}\theta) \tag{89}$$

Using the subscript 0 to denote the "initial" coordinates and momenta, we have

$$x_0 = A \tag{90}$$

$$y_0 = C \tag{91}$$

$$p_{x0} = m\omega B \tag{92}$$

$$p_{y0} = -m\omega\sqrt{\alpha}D \tag{93}$$

So,

$$\begin{aligned} x &= x_0\cos(\omega\theta) + \frac{p_{x0}}{m\omega}\sin(\omega\theta) \\ y &= y_0\cos(\omega\sqrt{\theta}) - \frac{p_{y0}}{m\omega\sqrt{\alpha}}\sin(\omega\sqrt{\alpha}\theta) \\ p_x &= -m\omega x_0\sin(\omega\theta) + p_{x0}\cos(\omega\theta) \\ p_y &= m\omega\sqrt{\alpha}y_0\sin(\omega\sqrt{\alpha}\theta) + p_{y0}\cos(\omega\sqrt{\alpha}\theta) \end{aligned}$$

Reverting to the notation in which x_0, p_{x0}, y_0, p_{y0} denote the original coordinates and X, Y, P_x, P_y denote the canonically transformed coordinates, we get the form of the canonical transformation as

$$X = x\cos(\omega\theta) + \frac{p_x}{m\omega}\sin(\omega\theta) \tag{94}$$

$$P_x = p_x\cos(\omega\theta) - m\omega x\sin(\omega\theta) \tag{95}$$

$$Y = y\cos(\omega\sqrt{\alpha}\theta) - \frac{p_y}{m\omega\sqrt{\alpha}}\sin(\omega\sqrt{\alpha}\theta) \tag{96}$$

$$P_y = p_y\cos(\omega\sqrt{\alpha}\theta) + m\omega\sqrt{\alpha}y\sin(\omega\sqrt{\alpha}\theta) \tag{97}$$

where θ is an arbitrary parameter, such that $\theta = 0$ corresponds to the untransformed coordinates. This canonical transformation is composed of two rotations in the 4-dimensional phase space (one involving X and P_x and the other involving Y and P_y), and its generator is the conserved quantity Δ.

Problem 4

Part (a)

$$F_2(q, P, t) = \left(q + \frac{1}{2}gt^2\right)(P - mgt) - \frac{P^2 t}{2m} \tag{98}$$

Now,

$$p = \frac{\partial F_2}{\partial q} = P - mgt \tag{99}$$

$$Q = \frac{\partial F_2}{\partial P} = q + \frac{1}{2}gt^2 - \frac{Pt}{m} = q + \frac{1}{2}gt^2 - \frac{pt}{m} - gt^2 \tag{100}$$

So, the canonical transformation is

$$P = p + mgt \tag{101}$$

$$Q = q - \frac{pt}{m} - \frac{1}{2}gt^2 \tag{102}$$

Part (b)

$$\{Q, Q\} = \{q - \frac{pt}{m} - \frac{1}{2}gt^2, q - \frac{pt}{m} - \frac{1}{2}gt^2\} = 0 \tag{103}$$

$$\{P, P\} = \{p + mgt, p + mgt\} = 0 \tag{104}$$

$$\begin{aligned}\{Q, P\} &= \{q - \frac{pt}{m} - \frac{1}{2}gt^2, p + mgt\} \\ &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= (1)(1) - \left(-\frac{t}{m}\right)(0) \\ &= 1\end{aligned} \tag{105}$$

So, the transformation satisfies the canonical Poisson Bracket relations.

Part (c)

The Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - mgq \tag{106}$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{107}$$

So the Hamiltonian is

$$H = p\dot{q} - L = \frac{p^2}{2m} + mgq \tag{108}$$

Now, $Q = q - \frac{pt}{m} - \frac{1}{2}gt^2$, so

$$\begin{aligned} \{Q, H\} &= \left\{q - \frac{pt}{m} - \frac{1}{2}gt^2, \frac{p^2}{2m} + mgq\right\} \\ &= \left\{q, \frac{p^2}{2m}\right\} - \left\{\frac{pt}{m}, mgq\right\} \\ &= \frac{1}{2m}\{q, p^2\} - gt\{p, q\} \\ &= \frac{p}{m} + gt \end{aligned} \tag{109}$$

Also

$$\frac{\partial Q}{\partial t} = -\frac{p}{m} - gt \tag{110}$$

So,

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, H\} = 0 \tag{111}$$

Also, $P = p + mgt$, so

$$\begin{aligned} \{P, H\} &= \{p + mgt, \frac{p^2}{2m} + mgq\} \\ &= mg\{p, q\} \\ &= -mg \end{aligned} \tag{112}$$

and

$$\frac{\partial P}{\partial t} = mg \tag{113}$$

So,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \{P, H\} = 0 \tag{114}$$

Part (d)

$$\begin{aligned}\frac{\partial F_2}{\partial t} &= gt(P-mgt)+\left(q+\frac{1}{2}gt^2\right)(-mg)-\frac{P^2}{2m}\\ &= Pgt-\frac{3}{2}mg^2t^2-mgq-\frac{P^2}{2m}\\ &= (p+mgt)gt-\frac{3}{2}mg^2t^2-mgq-\frac{(p+mgt)^2}{2m}\\ &= -\frac{p^2}{2m}-mgq-mg^2t^2 \end{aligned} \qquad (115)$$

So, the Hamiltonian associated with Q, P is

$$\begin{aligned}K &= H+\frac{\partial F_2}{\partial t}\\ &= \frac{p^2}{2m}+mgq-\frac{p^2}{2m}-mgq-mg^2t^2\\ &= -mg^2t^2\end{aligned} \qquad (116)$$

So, the Hamiltonian K is zero up to time-dependent constant term $-mg^2t^2$, but it is not a function of P and Q (which are constant with time, since $\{Q,H\}=\{P,H\}=0$ as shown above).

Part (e)

Q and P are conserved quantities, that equal the initial position and the initial momentum respectively. They are constant with time, as q and p vary:

$$\begin{aligned}q(t=0) &= Q\\ p(t=0) &= P\end{aligned}$$

Part (f)

$$\frac{\partial F_2}{\partial q} = P-mgt=p \qquad (117)$$

$$\frac{\partial F_2}{\partial t} = Pgt-\frac{mg^2t^2}{2}-mgq-\frac{P^2}{2m} \qquad (118)$$

$$\begin{aligned}H &= \frac{p^2}{2m}+mgq\\ &= \frac{1}{2m}\left(\frac{\partial F_2}{\partial q}\right)^2+mgq\end{aligned} \qquad (119)$$

So,

$$K = H+\frac{\partial F_2}{\partial t}=-mg^2t^2 \qquad \text{(as shown in part d)}$$

implies

$$H\left(q, \frac{\partial F_2}{\partial q}\right) + \frac{\partial F_2}{\partial t} = -mg^2t^2 \tag{120}$$

So, the Hamilton-Jacobi equation is satisfied, except for a time-dependent constant term appearing on the right hand side.

Part (g)

$$\begin{aligned}
f(Q,P,t) &= F_2(q(Q,P,t),P,t) \\
&= \left(q + \frac{1}{2}gt^2\right)(P - mgt) - \frac{P^2 t}{2m} \\
&= \left(Q + \frac{pt}{m} + gt^2\right)(P - mgt) - \frac{P^2 t}{2m} \\
&= \left(Q + \frac{(P-mgt)t}{m} + gt^2\right)(P - mgt) - \frac{P^2 t}{2m} \\
&= \left(Q + \frac{Pt}{m}\right)(P - mgt) - \frac{P^2 t}{2m} \\
&= QP + \frac{P^2 t}{2m} - Qmgt - gPt^2
\end{aligned} \tag{121}$$

So,

$$\frac{\partial f}{\partial t} = \frac{P^2}{2m} - mgQ - 2Pgt \tag{122}$$

Also, $p = m\dot{q} = P - mgt$. So,

$$\begin{aligned}
L(q,\dot{q}) &= \frac{p^2}{2m} - mgq \\
&= \frac{1}{2m}(P - mgt)^2 - mg\left(Q + \frac{Pt}{m} - \frac{1}{2}gt^2\right) \\
&= \frac{1}{2m}(P^2 + m^2g^2t^2 - 2Pmgt) - mgQ - Pgt + \frac{1}{2}mg^2t^2 \\
&= \frac{P^2}{2m} - 2Pgt - mgQ + mg^2t^2 \\
&= \frac{\partial f(Q,P,t)}{\partial t} + mg^2t^2
\end{aligned} \tag{123}$$

So, $L(q(Q,P,t), \dot{q}(Q,P,t)) = \frac{\partial f(Q,P,t)}{\partial t}$ up to a time-dependent term mg^2t^2.

Problem 5

The Hamilton-Jacobi equation, as expressed in the form

$$H(\boldsymbol{q}, \nabla S(\boldsymbol{q}, \boldsymbol{P})) + \frac{\partial S(\boldsymbol{q}, \boldsymbol{P})}{\partial t} = 0 \tag{124}$$

was obtained by constructing a generating function of the form

$$F = F_2(q, P, t) - Q_i P_i$$

where F_2 denotes a generic type-2 generating function. For such a choice of F, the Hamiltonian $K = H + \frac{\partial F}{\partial t}$ is zero.

Now, consider a type-3 generating function F_3 of the old momenta and the new coordinates, such that the Hamiltonian K is zero. Therefore,

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \tag{125}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \tag{126}$$

Now,

$$q_i = -\frac{\partial F_3}{\partial p_i} = -(\nabla_p F_3)_i \tag{127}$$

so,

$$H(q(Q, p), p, t) + \frac{\partial F_3}{\partial t}(\boldsymbol{Q}, \boldsymbol{p}, t) = 0 \tag{128}$$

where the old coordinates q have been expressed in terms of the old momenta and the new coordinates using equation (127). This is a PDE in $(n+1)$ variables $p_1, \ldots, p_n, t$. Let $\tilde{S}$ denote the solution of this PDE. Then, a solution of the form,

$$F_3 \equiv \tilde{S} = \tilde{S}(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_{n+1}; t) \tag{129}$$

where $Q_i = \alpha_i$ are the constants of motion (for $i = 1, \ldots, n$), is consistent with equation (125). Here the constant α_{n+1} must be a constant of integration, so the physically meaningful solution is of the form

$$\tilde{S} = \tilde{S}(p_1, \ldots, p_n; \alpha_1, \ldots \alpha_n; t) \tag{130}$$

So, in terms of $\tilde{S}$, equation (128) can be written as

$$H(-\boldsymbol{\nabla}_p \tilde{S}, \boldsymbol{p}, t) + \frac{\partial \tilde{S}}{\partial t}(\boldsymbol{Q}, \boldsymbol{p}, t) = 0 \tag{131}$$

which is of the desired form.