Theoretical Dynamics	December 09, 2010
Homework 11	
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Problem 1

Let ϕ, θ and ψ denote the Euler angles, and $\boldsymbol{\omega}_\phi = \dot{\boldsymbol{\phi}}$, $\boldsymbol{\omega}_\theta = \dot{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}_\psi = \dot{\boldsymbol{\psi}}$ denote their angular velocities. First, we express the components of these angular velocities along the body axes (x', y', z'), i.e. we find $(\omega_{x'}, \omega_{y'}, \omega_{z'})$. The full transformation can be depicted as

$$\underbrace{\left\{\begin{array}{c} \boldsymbol{\omega}_\phi = \dot{\phi} \\ \boldsymbol{\omega}_\theta = \dot{\theta} \\ \boldsymbol{\omega}_\psi = \dot{\psi} \end{array}\right\}}_{\text{body system}} \xrightarrow{T_1} \underbrace{\left\{\begin{array}{c} \boldsymbol{\omega}_{x'} \\ \boldsymbol{\omega}_{y'} \\ \boldsymbol{\omega}_{z'} \end{array}\right\}}_{\text{body system}} \tag{1}$$

The transformation, T_1, is a transformation from "body spherical coordinates" to "body Cartesian coordinates". Referring to diagram (c) below,

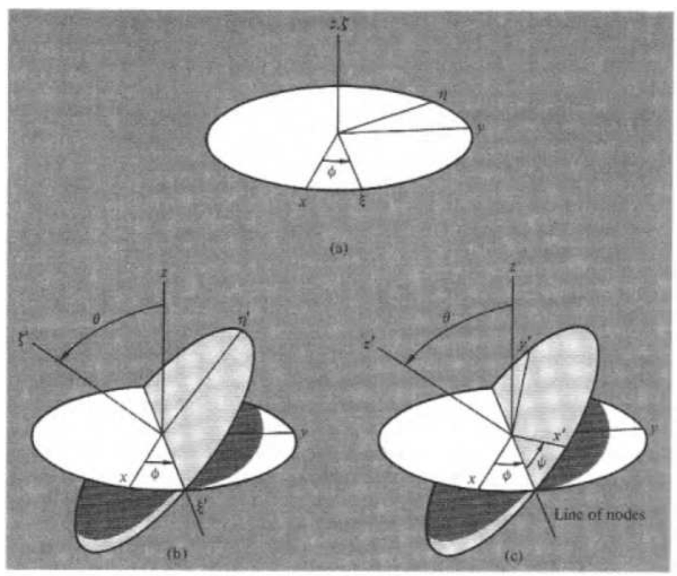


Figure defining the Euler angles. Figure courtesy Goldstein, Poole and Safko, *Classical Mechanics*, 3^{rd} Edn.

and using the fact that angular velocities for infinitesimal rotations are directed along the normal to the plane of rotation (consistent with the right hand screw rule), we can write

$$(\boldsymbol{\omega}_\phi)_{x'} = \dot{\phi}\sin\theta\sin\psi \tag{2}$$
$$(\boldsymbol{\omega}_\phi)_{y'} = \dot{\phi}\sin\theta\cos\psi \tag{3}$$
$$(\boldsymbol{\omega}_\phi)_{z'} = \dot{\phi}\cos\theta \tag{4}$$
$$(\boldsymbol{\omega}_\theta)_{x'} = \dot{\theta}\cos\psi \tag{5}$$
$$(\boldsymbol{\omega}_\theta)_{y'} = -\dot{\theta}\sin\theta \tag{6}$$
$$(\boldsymbol{\omega}_\theta)_{z'} = 0 \tag{7}$$
$$(\boldsymbol{\omega}_\psi)_{x'} = 0 \tag{8}$$
$$(\boldsymbol{\omega}_\psi)_{y'} = 0 \tag{9}$$
$$(\boldsymbol{\omega}_\psi)_{z'} = \dot{\psi} \tag{10}$$

Adding the corresponding components, we get

$$\omega_{x'}^{body} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{11}$$
$$\omega_{y'}^{body} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{12}$$
$$\omega_{z'}^{body} = \dot{\phi}\cos\theta + \dot{\psi} \tag{13}$$

Problem 2

The principal moments of inertia are $I_1 = I_2 = I_0$ and $I_3 = I_0/2$.

(a)

In this case, $\dot{\theta} = 0$ and $\dot{\phi} = 2\omega_0$. The angular momentum along the body-fixed z-axis is given to be constant and equal to $I_0\omega_0$. This implies

$$I_3\omega_3 = \frac{I_0}{2}\omega_3 = I_0\omega_0 \implies \omega_3 = 2\omega_0 \tag{14}$$

So, from Eqn. (13) above,

$$\dot{\phi}\cos\theta + \dot{\psi} = 2\omega_0 \tag{15}$$
$$\implies 2\omega_0\cos\theta + \dot{\psi} = 2\omega_0 \tag{16}$$

The condition for regular precession is

$$mgl = \dot{\phi}(I_3\omega_3 - I_1\dot{\phi}\cos\theta) \tag{17}$$
$$\implies mgl = 2\omega_0\left(\frac{I_0}{2}(2\omega_0) - I_0(2\omega_0)\cos\theta\right)$$
$$\implies mgl = 2I_0\omega_0^2(1 - 2\cos\theta)$$
$$\implies 1 = 2(1 - 2\cos\theta)$$
$$\implies \cos\theta = \frac{1}{4} \tag{18}$$

So, $\theta = \cos^{-1}\frac{1}{4}$.

(b)

The equation of motion is

$$\ddot{\theta} + \frac{1}{2}(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta - \omega_0^2 \sin\theta = 0 \tag{19}$$

Assuming that $\Delta\theta << \theta$, we can write

$$\begin{aligned} \sin(\theta + \Delta\theta) &= \sin\theta\cos\Delta\theta + \cos\theta\sin\Delta\theta \\ &\approx \sin\theta + \cos\theta\Delta\theta \\ &= \frac{\sqrt{15}}{4} + \frac{\Delta\theta}{4} \end{aligned} \tag{20}$$

$$\begin{aligned} \cos(\theta + \Delta\theta) &= \cos\theta\cos\Delta\theta - \sin\theta\sin\Delta\theta \\ &\approx \cos\theta - \sin\theta\Delta\theta \\ &\approx \frac{1}{4} - \frac{\sqrt{15}}{4}\Delta\theta \end{aligned} \tag{21}$$

Linearizing the equation of motion about $\theta_0 = \cos^{-1}\frac{1}{4}$, we get

$$\Delta\ddot{\theta} + \frac{\omega_0^2}{4}\Delta\theta = -\omega_0^2\frac{\sqrt{15}}{4} \tag{22}$$

from which it follows that the nutation frequency is $\frac{\omega_0}{2}$.

Problem 3

The equation of motion is

$$\ddot{x} = f(x) = -\lambda^2\omega_0^2 x \tag{23}$$

Let

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4 + \lambda^5 x_5 + \ldots \tag{24}$$

Substituting this into Eqn. (1) we get

$$\ddot{x}_0 + \lambda\ddot{x}_1 + \lambda^2\ddot{x}_2 + \lambda^3\ddot{x}_3 + \lambda^4\ddot{x}_4 + \lambda^5\ddot{x}_5 = -\lambda^2\omega_0^2 x_0 - \lambda^3\omega_0^2 x_1 - \lambda^4\omega_0^2 x_2 - \lambda^5\omega_0^2 x_3 \tag{25}$$

to order λ^5. Comparing coefficients of various powers of λ on both sides, we get

$$\ddot{x}_0 = 0 \tag{26}$$
$$\ddot{x}_1 = 0 \tag{27}$$
$$\ddot{x}_2 = -\omega_0^2 x_0 \tag{28}$$
$$\ddot{x}_3 = -\omega_0^2 x_1 \tag{29}$$
$$\ddot{x}_4 = -\omega_0^2 x_2 \tag{30}$$
$$\ddot{x}_5 = -\omega_0^2 x_3 \tag{31}$$

Using the boundary conditions given in the problem, the solutions to each of these second order differential equations can be determined successively. The results are:

$$x_0(t) = x(0) \tag{32}$$
$$x_1(t) = \dot{x}(0)t \tag{33}$$
$$x_2(t) = -\frac{\omega_0^2 x(0)t^2}{2} \tag{34}$$
$$x_3(t) = -\frac{\omega_0^2 \dot{x}(0)t^3}{6} \tag{35}$$
$$x_4(t) = \frac{\omega_0^4 x(0)t^4}{24} \tag{36}$$
$$x_5(t) = \frac{\omega_0^4 \dot{x}(0)t^5}{120} \tag{37}$$

(a)

The solution to order t^5 is:

$$\boxed{x(t) = x(0) + \dot{x}(0)t - \frac{\omega_0^2 x(0)t^2}{2} - \frac{\omega_0^2 \dot{x}(0)t^3}{6} + \frac{\omega_0^4 x(0)t^4}{24} + \frac{\omega_0^4 \dot{x}(0)t^5}{120}} \tag{38}$$

(b)

The exact solution is:

$$\begin{aligned} x_{ex}(t) &= x(0)\cos\omega_0 t + \frac{\dot{x}(0)}{\omega_0}\sin\omega_0 t & (39)\\ &= x(0)\left\{1 - \frac{\omega_0^2 t^2}{2} + \frac{\omega_0^4 t^4}{24} - \frac{\omega_0^6 t^6}{720} + \dots\right\} + \frac{\dot{x}(0)}{\omega_0}\left\{\omega_0 t - \frac{\omega_0^3 t^3}{6} + \frac{\omega_0^5 t^5}{120} + \dots\right\} & (40)\\ &= x(0) + \dot{x}(0)t - \frac{\omega_0^2 x(0)t^2}{2} - \frac{\omega_0^2 \dot{x}(0)t^3}{6} + \frac{\omega_0^4 x(0)t^4}{24} + \frac{\omega_0^4 \dot{x}(0)t^5}{120} + O(t^6) & (41) \end{aligned}$$

So to order t^5, the exact solution agrees with the perturbative solution.

Problem 4

The equations of motion are

$$\ddot{x} = -\frac{\lambda^2}{m}\frac{\partial V}{\partial x} \tag{42}$$
$$\ddot{y} = -\frac{\lambda^2}{m}\frac{\partial V}{\partial y} \tag{43}$$

Define

$$f(x,y) = -\frac{\partial V}{\partial x} \tag{44}$$
$$g(x,y) = -\frac{\partial V}{\partial y} \tag{45}$$

Further, let

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4 + \dots \tag{46}$$

$$y = y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4 + \dots \tag{47}$$

The functions $f(x, y)$ and $g(x, y)$ can be Taylor expanded about the point (x_0, y_0) (denoted by the subscript 0). For $f(x, y)$ the expansion is of the form

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_0 (y-y_0) + \left.\frac{\partial^2 f}{\partial x^2}\right|_0 \frac{(x-x_0)^2}{2} + \left.\frac{\partial^2 f}{\partial y^2}\right|_0 \frac{(y-y_0)^2}{2} \\ &+ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 \frac{(x-x_0)(y-y_0)}{2} + \left.\frac{\partial^2 f}{\partial y \partial x}\right|_0 \frac{(x-x_0)(y-y_0)}{2} + \text{ H.O.T.} \\ &= f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_0 (y-y_0) + \left.\frac{\partial^2 f}{\partial x^2}\right|_0 \frac{(x-x_0)^2}{2} + \left.\frac{\partial^2 f}{\partial y^2}\right|_0 \frac{(y-y_0)^2}{2} \\ &+ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 (x-x_0)(y-y_0) + \text{ H.O.T.} \end{aligned} \tag{48}$$

Plugging this into the differential equation for x, we get (retaining terms up to second order in the partial derivatives)

$$\begin{aligned} \ddot{x}_0 + \lambda \ddot{x}_1 + \lambda^2 \ddot{x}_2 + \lambda^3 \ddot{x}_3 + \lambda^4 \ddot{x}_4 = -\frac{\lambda^2}{m} \Bigg[& f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_0 (y-y_0) \\ & + \left.\frac{\partial^2 f}{\partial x^2}\right|_0 \frac{(x-x_0)^2}{2} + \left.\frac{\partial^2 f}{\partial y^2}\right|_0 \frac{(y-y_0)^2}{2} + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 (x-x_0)(y-y_0) \Bigg] \end{aligned} \tag{49}$$

$$\begin{aligned} = -\frac{\lambda^2}{m} \Bigg[& f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 (\lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4) \\ & + \left.\frac{\partial f}{\partial y}\right|_0 (\lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4) + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_0 (\lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4)^2 \\ & + \frac{1}{2} \left.\frac{\partial^2 f}{\partial y^2}\right|_0 (\lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4)^2 \\ & + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 (\lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \lambda^4 x_4)(\lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4) \Bigg] \end{aligned} \tag{50}$$

Equating the coefficients of various powers of λ on both sides, we get

$$\ddot{x}_0 = 0 \tag{51}$$

$$\ddot{x}_1 = 0 \tag{52}$$

$$\ddot{x}_2 = -\frac{1}{m} f(x_0, y_0) \tag{53}$$

$$\ddot{x}_3 = -\frac{1}{m} \left[\left.\frac{\partial f}{\partial x}\right|_0 x_1 + \left.\frac{\partial f}{\partial y}\right|_0 y_1 \right] \tag{54}$$

$$\ddot{x}_4 = -\frac{1}{m} \left[\left.\frac{\partial f}{\partial x}\right|_0 x_2 + \left.\frac{\partial f}{\partial y}\right|_0 y_2 + \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_0 x_1^2 + \frac{1}{2} \left.\frac{\partial^2 f}{\partial y^2}\right|_0 y_1^2 \right] \tag{55}$$

Following a similar procedure for the differential equation for y, we get

$$\ddot{y}_0 = 0 \tag{56}$$
$$\ddot{y}_1 = 0 \tag{57}$$
$$\ddot{y}_2 = -\frac{1}{m} g(x_0, y_0) \tag{58}$$
$$\ddot{y}_3 = -\frac{1}{m}\left[\left.\frac{\partial g}{\partial x}\right|_0 x_1 + \left.\frac{\partial g}{\partial y}\right|_0 y_1\right] \tag{59}$$
$$\ddot{y}_4 = -\frac{1}{m}\left[\left.\frac{\partial g}{\partial x}\right|_0 x_2 + \left.\frac{\partial g}{\partial y}\right|_0 y_2 + \frac{1}{2}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 x_1^2 + \frac{1}{2}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 y_1^2\right] \tag{60}$$

The solutions are

$$x_0(t) = c_1 t + c_2, \qquad y_0(t) = d_1 t + d_2 \tag{61}$$
$$x_1(t) = c_3 t + c_4, \qquad y_1(t) = d_3 t + d_4 \tag{62}$$
$$x_2(t) = -\frac{1}{2m} f(x_0, y_0) t^2 + c_5 t + c_6 \tag{63}$$
$$y_2(t) = -\frac{1}{2m} g(x_0, y_0) t^2 + d_5 t + d_6 \tag{64}$$
$$x_3(t) = -\frac{1}{2m}\left[c_4 \left.\frac{\partial f}{\partial x}\right|_0 + d_4 \left.\frac{\partial f}{\partial y}\right|_0\right] t^2 - \frac{1}{6m}\left[c_3 \left.\frac{\partial f}{\partial x}\right|_0 + d_3 \left.\frac{\partial f}{\partial y}\right|_0\right] t^3 + c_7 t + c_8 \tag{65}$$
$$y_3(t) = -\frac{1}{2m}\left[c_4 \left.\frac{\partial g}{\partial x}\right|_0 + d_4 \left.\frac{\partial g}{\partial y}\right|_0\right] t^2 - \frac{1}{6m}\left[c_3 \left.\frac{\partial g}{\partial x}\right|_0 + d_3 \left.\frac{\partial g}{\partial y}\right|_0\right] t^3 + d_7 t + d_8 \tag{66}$$
$$\begin{aligned} x_4(t) = &\left\{\frac{1}{2m^2}\left.\frac{\partial f}{\partial x}\right|_0 f(x_0, y_0) + \frac{1}{2m^2}\left.\frac{\partial f}{\partial y}\right|_0 g(x_0, y_0) - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 c_3^2 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_3^2 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 c_3 d_3\right\}\frac{t^4}{12} \\ &+ \left\{-\frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 c_5 - \frac{1}{m}\left.\frac{\partial f}{\partial y}\right|_0 d_5 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 c_3 c_4 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_3 d_4 - \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 (c_3 d_4 + c_4 d_3)\right\}\frac{t^3}{6} \\ &+ \left\{-\frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 c_6 - \frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 d_6 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_4 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 c_4 d_4\right\}\frac{t^2}{2} + c_9 t + c_{10} \end{aligned} \tag{67}$$
$$\begin{aligned} y_4(t) = &\left\{\frac{1}{2m^2}\left.\frac{\partial g}{\partial x}\right|_0 g(x_0, y_0) + \frac{1}{2m^2}\left.\frac{\partial g}{\partial y}\right|_0 f(x_0, y_0) - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 c_3^2 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_3^2 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 c_3 d_3\right\}\frac{t^4}{12} \\ &+ \left\{-\frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 c_5 - \frac{1}{m}\left.\frac{\partial g}{\partial y}\right|_0 d_5 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 c_3 c_4 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_3 d_4 - \left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 (c_3 d_4 + c_4 d_3)\right\}\frac{t^3}{6} \\ &+ \left\{-\frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 c_6 - \frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 d_6 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_4 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 c_4 d_4\right\}\frac{t^2}{2} + d_9 t + d_{10} \end{aligned} \tag{68}$$

So, the general solution for $x(t)$ is $x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$, that is,

$$x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 \tag{69}$$

where

$$\alpha_0 = c_2 + c_4 + c_6 + c_8 + c_{10} = x(0) \tag{70}$$

$$\alpha_1 = c_1 + c_3 + c_5 + c_7 + c_9 = \dot{x}(0) \tag{71}$$

$$\begin{aligned}\alpha_2 = &-\frac{1}{2m} f(x_0, y_0) - \frac{1}{2m}\left[c_4 \left.\frac{\partial f}{\partial x}\right|_0 + d_4 \left.\frac{\partial f}{\partial y}\right|_0\right] \\ &+ \frac{1}{2}\left\{-\frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 c_6 - \frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 d_6 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_4 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 c_4 d_4\right\}\end{aligned} \tag{72}$$

$$\alpha_3 = \frac{1}{6}\left\{-\frac{1}{m}\left.\frac{\partial f}{\partial x}\right|_0 c_5 - \frac{1}{m}\left.\frac{\partial f}{\partial y}\right|_0 d_5 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 c_3 c_4 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_3 d_4 - \left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 (c_3 d_4 + c_4 d_3)\right\} \tag{73}$$

$$\alpha_4 = \frac{1}{12}\left\{\frac{1}{2m^2}\left.\frac{\partial f}{\partial x}\right|_0 f(x_0, y_0) + \frac{1}{2m^2}\left.\frac{\partial f}{\partial y}\right|_0 g(x_0, y_0) - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial x^2}\right|_0 c_3^2 - \frac{1}{2m}\left.\frac{\partial^2 f}{\partial y^2}\right|_0 d_3^2 - \frac{1}{m}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_0 c_3 d_3\right\} \tag{74}$$

Similarly, the general solution for $y(t)$ is

$$y(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 \tag{75}$$

where

$$\beta_0 = d_2 + d_4 + d_6 + d_8 + d_{10} = y(0) \tag{76}$$

$$\beta_1 = d_1 + d_3 + d_5 + d_7 + d_9 = \dot{y}(0) \tag{77}$$

$$\begin{aligned}\beta_2 = &-\frac{1}{2m} g(x_0, y_0) - \frac{1}{2m}\left[c_4 \left.\frac{\partial g}{\partial x}\right|_0 + d_4 \left.\frac{\partial g}{\partial y}\right|_0\right] \\ &+ \frac{1}{2}\left\{-\frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 c_6 - \frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 d_6 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_4 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 c_4 d_4\right\}\end{aligned} \tag{78}$$

$$\beta_3 = \frac{1}{6}\left\{-\frac{1}{m}\left.\frac{\partial g}{\partial x}\right|_0 c_5 - \frac{1}{m}\left.\frac{\partial g}{\partial y}\right|_0 d_5 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 c_3 c_4 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_3 d_4 - \left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 (c_3 d_4 + c_4 d_3)\right\} \tag{79}$$

$$\beta_4 = \frac{1}{12}\left\{\frac{1}{2m^2}\left.\frac{\partial g}{\partial x}\right|_0 g(x_0, y_0) + \frac{1}{2m^2}\left.\frac{\partial g}{\partial y}\right|_0 f(x_0, y_0) - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial x^2}\right|_0 c_3^2 - \frac{1}{2m}\left.\frac{\partial^2 g}{\partial y^2}\right|_0 d_3^2 - \frac{1}{m}\left.\frac{\partial^2 g}{\partial x \partial y}\right|_0 c_3 d_3\right\} \tag{80}$$

From the differential equations, we have

$$\ddot{x}(t=0) = -\frac{f(x_0, y_0)}{m} \implies \alpha_2 = -\frac{f(x_0, y_0)}{2m} \tag{81}$$

$$\ddot{y}(t=0) = -\frac{g(x_0, y_0)}{m} \implies \beta_2 = -\frac{g(x_0, y_0)}{2m} \tag{82}$$

and

$$\left.\frac{d^3 x}{dt^3}\right|_{t=0} = -\frac{1}{m}\left[\left.\frac{\partial f}{\partial x}\right|_0 \dot{x}(0) + \left.\frac{\partial f}{\partial y}\right|_0 \dot{y}(0)\right] = 6\alpha_3 \tag{83}$$

$$\left.\frac{d^3 y}{dt^3}\right|_{t=0} = -\frac{1}{m}\left[\left.\frac{\partial g}{\partial x}\right|_0 \dot{x}(0) + \left.\frac{\partial g}{\partial y}\right|_0 \dot{y}(0)\right] = 6\beta_3 \tag{84}$$

$$\left.\frac{d^4x}{dt^4}\right|_{t=0} = -\frac{1}{m}\left[\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\right)_{t=0} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\right)_{t=0}\dot{x}(0) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}\right)_{t=0}\dot{y}(0)\right]$$
$$= 24\alpha_4 \tag{85}$$
$$\left.\frac{d^4y}{dt^4}\right|_{t=0} = -\frac{1}{m}\left[\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}\right)_{t=0} + \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}\right)_{t=0}\dot{x}(0) + \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}\right)_{t=0}\dot{y}(0)\right]$$
$$= 24\beta_4 \tag{86}$$

In this manner, we can determine the α_i's and β_i's from the boundary conditions, and hence determine $x(t)$ and $y(t)$ to order t^4.

Problem 5

The equation of motion is

$$\ddot{\theta} + \frac{\lambda V_0}{I}\sin\theta = 0 \tag{87}$$

Now,

$$\theta(t) = \omega_0 t + \lambda\theta_1 + \lambda^2\theta_2 + \dots \tag{88}$$

As we are interested only in a solution to $O(\lambda^2)$,

$$\begin{aligned}\sin(\theta) &\approx \sin(\omega_0 + \lambda\theta_1) = \sin(\omega_0 t)\cos(\lambda\theta_1) + \sin(\lambda\theta_1)\cos(\omega_0 t)\\ &\approx \sin(\omega_0 t) + \lambda\theta_1\cos(\omega_0 t)\end{aligned} \tag{89}$$

Substituting

$$\lambda\ddot{\theta}_1 + \lambda^2\ddot{\theta}_2 + \frac{\lambda V_0}{I}(\sin(\omega_0 t) + \lambda\theta_1\cos(\omega_0 t)) = 0 \tag{90}$$

Equating coefficients of like powers of λ on both sides, we get

$$\ddot{\theta}_1 = -\frac{V_0}{I}\sin(\omega_0 t) \tag{91}$$
$$\ddot{\theta}_2 = -\frac{V_0}{I}\theta_1(t)\cos(\omega_0 t) \tag{92}$$

Using the given boundary conditions, $\theta_i(0) = 0$ and $\dot{\theta}_i(0) = 0$, these equations can be solved analytically, to yield the solutions

$$\theta_1(t) = \frac{V_0}{I\omega_0^2}[\sin(\omega_0 t) - \omega_0 t] \tag{93}$$
$$\theta_2(t) = \frac{V_0^2}{I^2\omega_0^2}\left[\frac{\sin(2\omega_0 t)}{8} - \omega_0 t\cos(\omega_0 t) + 2\sin(\omega_0 t) - \frac{5\omega_0 t}{4}\right] \tag{94}$$

Problem 6

(a)

The solution $\theta(t)$ for an anharmonic oscillator must satisfy the differential equation

$$\ddot{\theta} = -\frac{V_0}{I}\sin\theta \tag{95}$$

The solution $\theta(t)$ must be periodic with some period T such that if $\theta(t=0)=0$ then $\theta(t=T)=2\pi$. Since the energy,

$$E = \frac{1}{2}I\dot{\theta}^2 - V_0\cos\theta \tag{96}$$

is conserved, we must have $\dot{\theta}(t=0)=\dot{\theta}(t=2\pi)$. That is, the angular speed after one period will be identical to that at $t=0$. If V_0 is negligibly small, then the solution is $\theta_0=\omega_0 t$ where $\omega_0=\frac{2\pi}{T}$, with the understanding that $\theta(t+T)=\theta(t)$ (that is, the angles are measured modulo 2π). This ensures periodicity. If V_0 is not negligibly small, then this solution still predicts the correct angular position at times $t=nT$ where n is an integer, because the potential energy itself is a periodic function. Deviations from this solution will occur between two successive periods, i.e. $nT<t<(n+1)T$, due to the variation of the potential energy term, and will vanish at $t=nT$.

So, the deviations can be expanded in a Fourier sine series of the form $\sum_n c_n\sin(n\omega t)$, where $n\omega T = 2\pi m$ where n and m are integers, so that the argument of the sine term ensures that the deviations vanish at integral multiples of the period T. That is, the solution can be written as

$$\boxed{\theta(t) = \omega t + \sum_{n=1}^{\infty} c_n \sin(n\omega t)} \tag{97}$$

In general, $\omega\neq\omega_0$. This is because of the fact that in the anharmonic regime, we cannot approximate $\sin\theta$ as θ and so the effect of the gravitational potential V_0 is to slow the pendulum down so that it takes longer to complete one cycle than it would if the oscillation frequency were ω_0. Therefore, $\omega<\omega_0$. Note that the existence of higher Fourier modes emphasizes the anharmonicity of the solution.

(b)

We have

$$\omega = \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \ldots \tag{98}$$

$$c_n = \lambda^n(c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots) \tag{99}$$

So, up to second order in λ, we can write

$$\theta(t) = (\omega_0+\lambda\omega_1+\lambda^2\omega_2)t + \lambda(c_1^{(0)}+\lambda c_1^{(1)})\sin[(\omega_0+\lambda\omega_1+\lambda^2\omega_2)t] + \lambda^2 c_2^{(0)}\sin[2(\omega_0+\lambda\omega_1+\lambda^2\omega_2)t] \tag{100}$$

Since the sine terms are premultiplied by λ, they can be approximated as

$$\sin[(\omega_0 + \lambda\omega_1 + \lambda^2\omega_2)t] \approx \sin[(\omega_0 + \lambda\omega_1)t] = \sin(\omega_0 t)\cos(\lambda\omega_1 t) + \cos(\omega_0 t)\sin(\lambda\omega_1 t)$$
$$\approx \sin(\omega_0 t) + \lambda\omega_1 t\cos(\omega_0 t) \tag{101}$$
$$\sin[2(\omega_0 + \lambda\omega_1 + \lambda^2\omega_2)t] \approx \sin[2(\omega_0 + \lambda\omega_1)t] \approx \sin(2\omega_0 t) \tag{102}$$

as we are only interested in terms to order λ^2. Substituting these into the expression for $\theta(t)$ and rearranging, we get

$$\theta(t) = \omega_0 t + \lambda[\omega_1 t + c_1^{(0)}\sin(\omega_0 t)] + \lambda^2[\omega_2 t + c_1^{(0)}\omega_1 t\cos(\omega_0 t) + c_1^{(1)}\sin(\omega_0 t) + c_2^{(0)}\sin(2\omega_0 t)] \tag{103}$$

Comparing Eqn. (103) with Eqns. (88), (93) and (94) we get

$$\omega_1 = -\frac{V_0}{I\omega_0^2}\omega_0, \qquad \omega_2 = -\frac{V_0^2}{I^2\omega_0^4} \times \frac{5}{4}\omega_0 \tag{104}$$
$$c_1^{(0)} = \frac{V_0}{I\omega_0^2}, \qquad c_1^{(1)} = \frac{V_0^2}{I^2\omega_0^4} \times 2 \tag{105}$$
$$c_2^{(0)} = \frac{V_0^2}{I^2\omega_0^4} \times \frac{1}{8} \tag{106}$$

Let $\alpha = \frac{V_0}{I\omega_0^2}$. Then,

$$\omega = \omega_0\left(1 - \alpha - \frac{5}{4}\alpha^2\right) \tag{107}$$
$$\theta(t) = \omega t + (\alpha + 2\alpha^2)\sin(\omega t) + \frac{1}{8}\alpha^2\sin(2\omega t) \tag{108}$$

Eqn. (107) confirms tha $\omega < \omega_0$ as reasoned in part (a).

Problem 7

The energy is

$$E = \frac{1}{2}I\dot{\theta}^2 - V_0\cos\theta \tag{109}$$

At $t = 0$, $\dot{\theta}(t) = \omega_0$ and $\theta = 0$, so

$$\frac{1}{2}I\omega_0^2 - V_0 = 10V_0 \tag{110}$$

implies that

$$\boxed{\omega_0 = \sqrt{\frac{22V_0}{I}}} \tag{111}$$

So,

$$\boxed{\alpha = \frac{V_0}{I\omega_0^2} = \frac{1}{22}} \tag{112}$$

We assume that $V_0 = 1$ and $I = 1$ in suitable units, so that $\omega_0 = \sqrt{22}\,rad/s$ and $\alpha = 1/22$. The plots for problems 5 and 6 are:

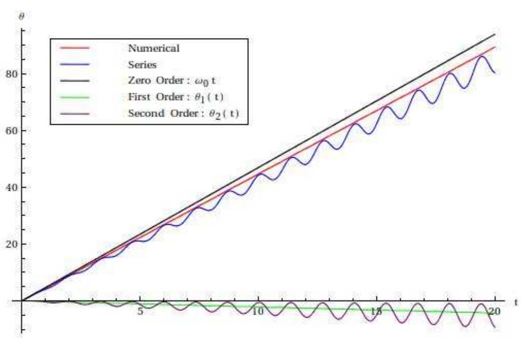


Figure for problem 5

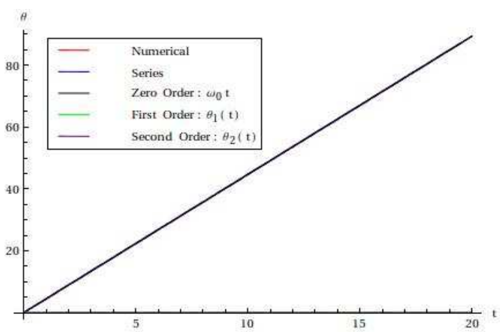
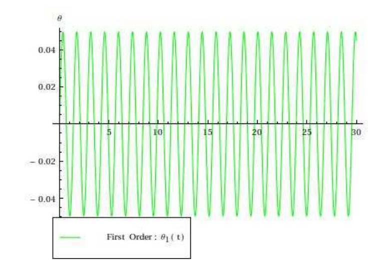
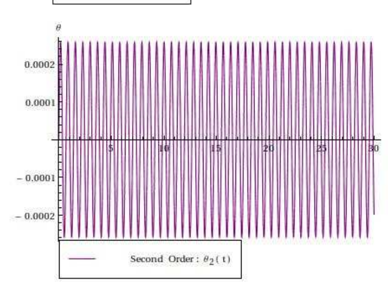


Figure for problem 6

In the plot for problem 6, the large magnitudes of the numerical and series solution and zero-th order terms mask out the first and second order oscillatory terms, which are shown separately below:





For detailed calculations, and plots of all the terms, please refer to the attached Mathematica notebook, `problem-7.nb`. From the plots, it follows that the solutions are very close for large values of t.

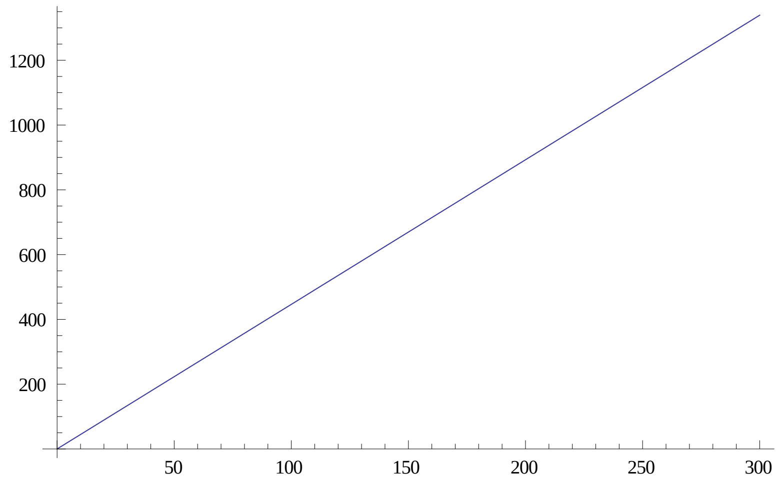
```
Clear["Global`*"];
```

Problem 5 (Numerical Solution): The differential equation is $\theta'' + \frac{V_0}{I}\sin\theta = 0$. Let $k = \frac{V_0}{I}$. We solve for the case k = 1.

```
ifun = First[y /.
   NDSolve[{y''[t] + 1 * Sin[y[t]] == 0, y[0] == 0, y'[0] == Sqrt[22 * 1]}, y, {t, 0, 300}]]
```

```
InterpolatingFunction[{{0., 300.}}, <>]
```

```
Plot[ifun[t], {t, 0, 300}]
```



Series Solution obtained in Problem 5.

$$f1[\omega\_, t\_] = \frac{V0}{J * \omega^2} (Sin[\omega\, t] - \omega\, t);$$

$$f2[\omega\_, t\_] = \frac{V0}{J * J * \omega^2} \left(\frac{Sin[2\, \omega\, t]}{8} - \omega\, t\, Cos[\omega\, t] + 2\, Sin[\omega\, t] - \frac{5\, \omega\, t}{4}\right);$$

$$s[\omega\_, t\_] = \omega * t + \frac{V0}{J * \omega^2} (Sin[\omega\, t] - \omega\, t) + \frac{V0}{J * J * \omega^2} \left(\frac{Sin[2\, \omega\, t]}{8} - \omega\, t\, Cos[\omega\, t] + 2\, Sin[\omega\, t] - \frac{5\, \omega\, t}{4}\right);$$

```
f1s[t_] = f1[Sqrt[22], t] /. {V0 → 1, J → 1}
```

$$\frac{1}{22}\left(-\sqrt{22}\, t + Sin\left[\sqrt{22}\, t\right]\right)$$

```
f2s[t_] = f2[Sqrt[22], t] /. {V0 → 1, J → 1}
```

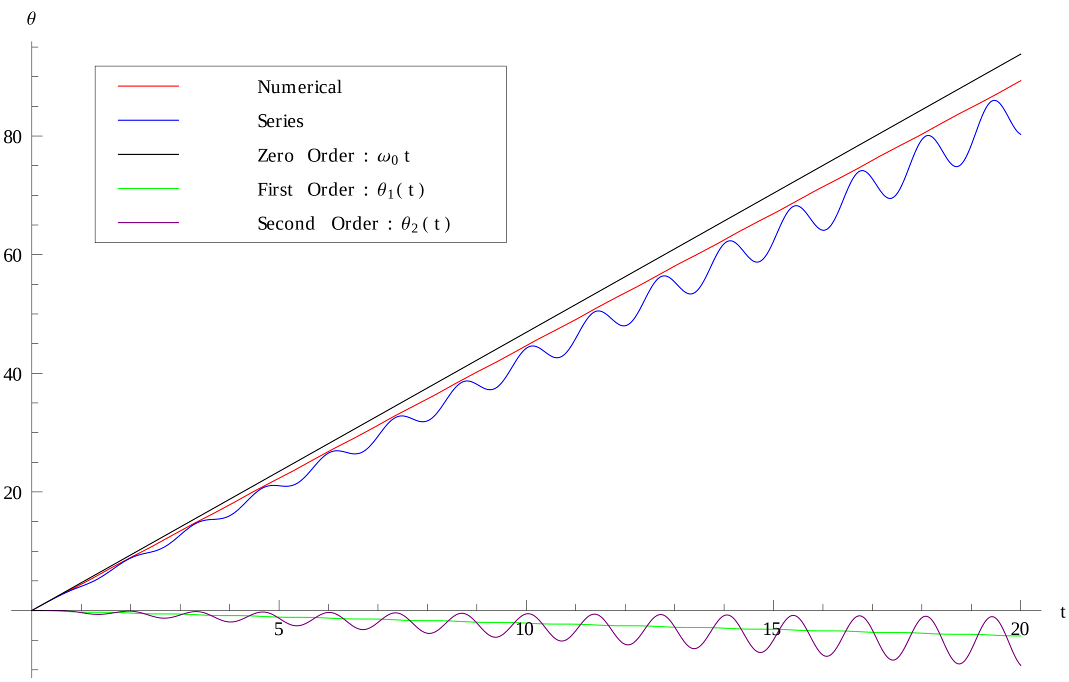
$$\frac{1}{22}\left(-\frac{5}{2}\sqrt{\frac{11}{2}}\, t - \sqrt{22}\, t\, Cos\left[\sqrt{22}\, t\right] + 2\, Sin\left[\sqrt{22}\, t\right] + \frac{1}{8}\, Sin\left[2\sqrt{22}\, t\right]\right)$$

```
sol[t_] = s[Sqrt[22], t] /. {V0 → 1, J → 1} // FullSimplify
```

$$\frac{1}{176}\left(2\sqrt{22}\ t\left(79 - 4\ \mathrm{Cos}\left[\sqrt{22}\ t\right]\right) + 24\ \mathrm{Sin}\left[\sqrt{22}\ t\right] + \mathrm{Sin}\left[2\sqrt{22}\ t\right]\right)$$

```
<< PlotLegends`

Plot[{ifun[t], sol[t], Sqrt[22] * t, f1s[t], f2s[t]}, {t, 0, 20},
 AxesLabel → {"t", "θ"}, PlotLegend → {"Numerical", "Series",
   "Zero Order: ω0 t", "First Order: θ1(t)", "Second Order: θ2(t)"},
 LegendShadow → None, PlotStyle → {Red, Blue, Black, Green, Purple}]
```



Series Solution obtained in Problem 6.

```
ω6 = ω0 * (1 - α - 5 α^2/4);

ω6 /. {α → 1/22, ω0 → Sqrt[22]} // FullSimplify
```

$$\frac{1843}{88\sqrt{22}}$$

```
term1[t_] = (α + 2 α^2) Sin[ω6 * t] /. {α → 1/22, ω0 → Sqrt[22]}
```

$$\frac{6}{121}\ \mathrm{Sin}\left[\frac{1843\ t}{88\sqrt{22}}\right]$$

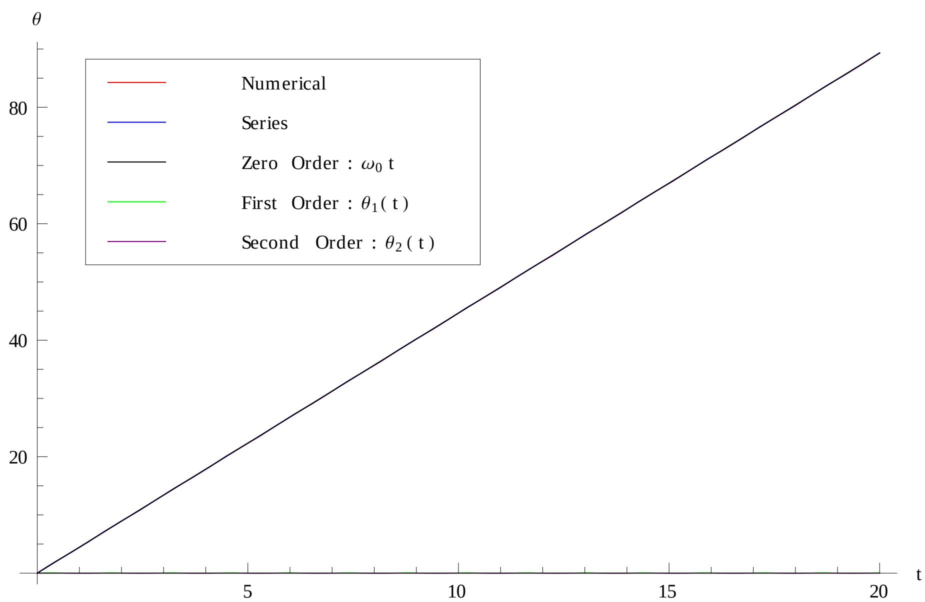
```
term2[t_] = 1/8 α^2 Sin[2 ω6 * t] /. {α → 1 / 22, ω0 → Sqrt[22]}
```

$$\frac{\mathrm{Sin}\left[\frac{1843\, t}{44\sqrt{22}}\right]}{3872}$$

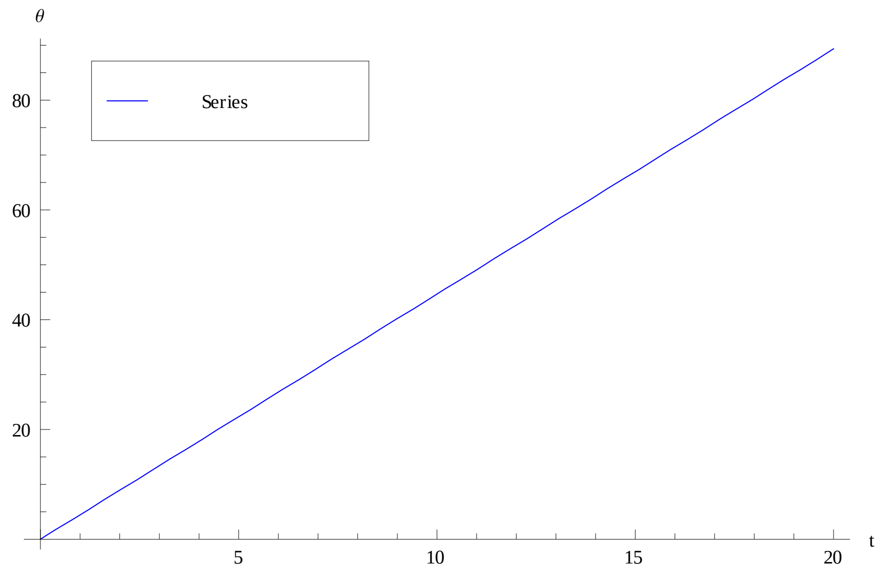
```
sol6[t_] = 1843/(88 Sqrt[22]) * t + term1[t] + term2[t]
```

$$\frac{1843\, t}{88\sqrt{22}} + \frac{6}{121}\mathrm{Sin}\left[\frac{1843\, t}{88\sqrt{22}}\right] + \frac{\mathrm{Sin}\left[\frac{1843\, t}{44\sqrt{22}}\right]}{3872}$$

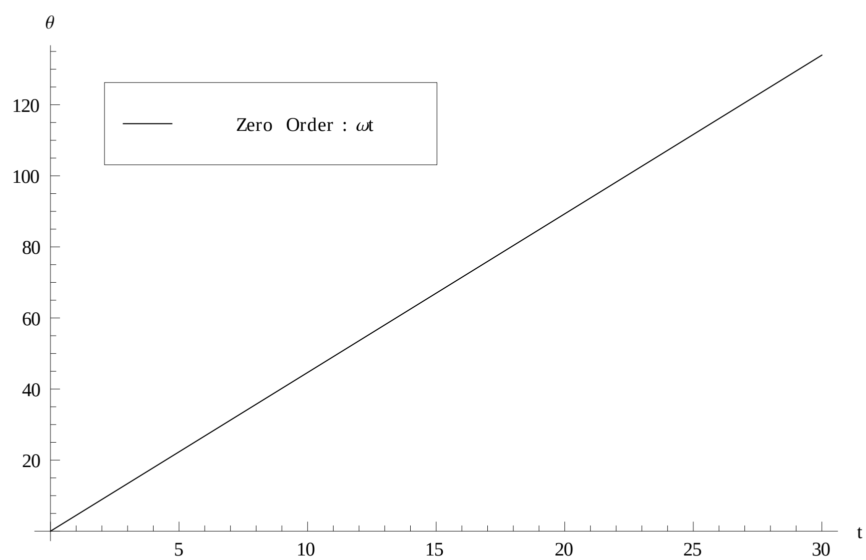
```
Plot[{ifun[t], sol6[t], 1843/(88 Sqrt[22]) * t, term1[t], term2[t]},
 {t, 0, 20}, AxesLabel → {"t", "θ"}, PlotLegend → {"Numerical", "Series",
   "Zero Order: ω0 t", "First Order: θ1(t)", "Second Order: θ2(t)"},
 LegendShadow → None, PlotStyle → {Red, Blue, Black, Green, Purple}]
```



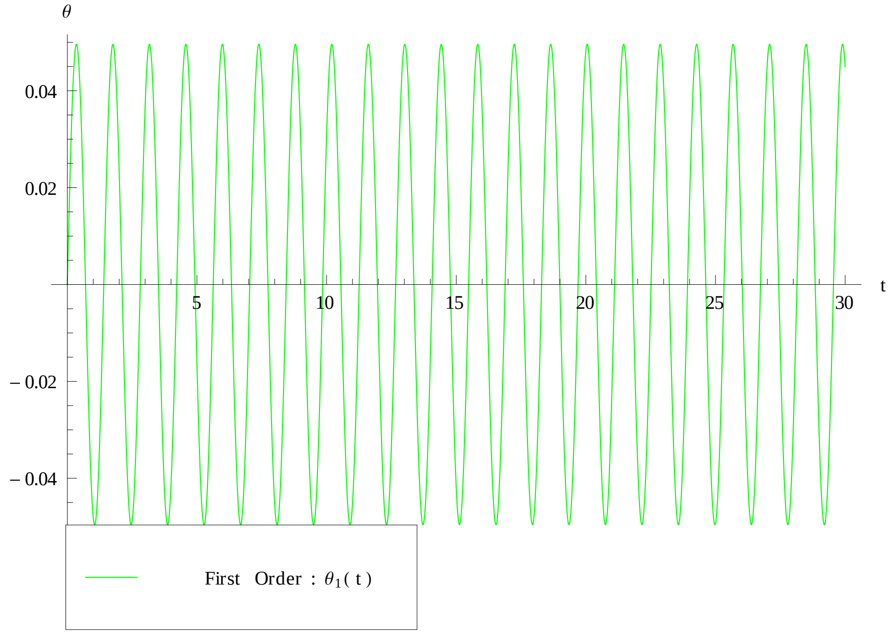

```
Plot[sol6[t], {t, 0, 20}, AxesLabel → {"t", "θ"},
 PlotLegend → {"Series"}, LegendShadow → None, PlotStyle → {Blue}]
```




```
Plot[1843/(88 Sqrt[22]) * t, {t, 0, 30}, AxesLabel → {"t", "θ"},
 PlotLegend → {"Zero Order: ωt"}, LegendShadow → None, PlotStyle → {Black}]
```




```
Plot[term1[t], {t, 0, 30}, AxesLabel → {"t", "θ"},
 PlotLegend → {"First Order: θ1(t)"}, LegendShadow → None, PlotStyle → {Green}]
```




```
Plot[term2[t], {t, 0, 30}, AxesLabel → {"t", "θ"},
 PlotLegend → {"Second  Order :  θ2(t)"}, LegendShadow → None, PlotStyle → {Purple}]
```