Physics 601 Homework 10---Due Friday November 12

Goldstein---4.22, 4.24, 4.25
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1. This problem concerns rotations about the z axis.
   a. Show that the rotation about the z axis: $\vec{\vec{R}} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ can be written as $\vec{\vec{R}} = \exp\left(-\theta M_z\right)$ where $M^z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
   b. Show that $\cos(\vartheta) = \frac{1}{2}\mathrm{tr}(\vec{\vec{R}}) - \frac{1}{2}$
2. There is a general theorem by Euler that any rotation matrix can be represented as a rotation about one given axis. Thus by analogy to problem 3a. it can be written as $\vec{\vec{R}} = \exp\left(-\Phi\hat{n}\cdot\vec{\vec{\vec{M}}}\right) = \exp\left(-\Phi\left(n_x\vec{\vec{M}}_x + n_y\vec{\vec{M}}_y + n_z\vec{\vec{M}}_z\right)\right)$ where $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is the unit vector specify the axis of rotation and Φ is the angle specifying the rotation. The purpose of this problem is to find the explicit of the rotation matrix for such a rotation. As a first step note that $\hat{n}$ is completely specified by a polar angle θ and azimuthal angle ϕ. Define $\vec{\vec{R}}_{\hat{n}} \equiv \vec{\vec{R}}_z(\phi)\vec{\vec{R}}_y(\theta)$.
   a. As a first step show that $\hat{n}\cdot\vec{\vec{\vec{M}}} = \vec{\vec{R}}_{\hat{n}}\vec{\vec{M}}_z\vec{\vec{R}}_{\hat{n}}^T$.
   b. Show that $\vec{\vec{R}} = \exp(-\Phi\,\hat{n}\cdot\vec{\vec{\vec{M}}}) = \vec{\vec{R}}_{\hat{n}}\exp(-\Phi\vec{\vec{M}}^z)\vec{\vec{R}}_{\hat{n}}^T$
   c. Express the nine matrix elements of $\vec{\vec{R}}$ in terms of the polar and azimuthal angles defining $\hat{n}$ and the rotation angle Φ.

   Not that this description of the rotation matrices is alternative parameterization to the Euler angles.

3. For the general rotation of the form given in 2:
   a. Show that $\hat{n}$ is an eignevector of $\vec{\vec{R}}$ with eigenvalue one.
   b. Show that $\cos(\Phi) = \frac{1}{2}\mathrm{Tr}(\vec{\vec{R}}) - \frac{1}{2}$.
4. Suppose we have a rotation specified by the Euler angles $\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right)$ (that is $\vec{\vec{R}} = \vec{\vec{R}}_z\left(\frac{\pi}{3}\right)\vec{\vec{R}}_x\left(\frac{\pi}{3}\right)\vec{\vec{R}}_z\left(\frac{\pi}{3}\right)$). Find the angle of rotation about the single axis and the axis $\hat{n}$.