Physics 601 Homework 12---Due Friday Dec. 10

1. Last week, I had you consider the case of a physical pendulum in the regime the high energy regime where the kinetic energy is much large then the potential. This time I want to consider the opposite regime---extremely low energies. Again the Lagrangian is given by $L = \frac{1}{2} I\dot{\theta}^2 + V_0 \cos(\theta)$ where *I* is the moment of inertia and V_0 is the maximum value of the potential energy (i.e. *m g L*) where L is length from the pivot point to the center of mass. The problem under consideration is this: suppose that at t=0 the system is at rest with $\theta = \theta_i$. I wish to develop a systematic power counting scheme for this problem.

   Consider the following power counting scheme in terms of λ: $L = \frac{1}{2} I\dot{\theta}^2 + \frac{V_0 \cos(\lambda^{\frac{1}{2}}\theta)}{\lambda}$

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   a. Assuming this system is at low energy and is θ small and one can expand out the cosine. Show that this is sensible in that leading order term with the harmonic term in leading order, a quartic term as a perturbation and a sextic term as a smaller perturbation and so on.

   b. One can write $\theta = \theta_0 + \lambda\theta_1 + \lambda^2\theta_2 + \ldots$. Find explicitly the form for θ_1 and θ_2.

2. The approximate solution found in problem with naïve perturbation theory does not impose periodicity. However on general grounds we expect that $\theta(t) = \sum_n c_n \cos((2n+1)\omega t)$. Match this form onto the perturbative solution using the power counting scheme $\omega = \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \ldots,$ to compute the frequency as a function of amplitude up to the order computed. Show that this gives $\omega(\theta_0)$ as a series in θ_0 with accurate terms up to 2nd order.

3. Consider a particle of mass *m* and charge *q* particle moving in a spatial constant (but time varying magnetic field). For the sake of simplicity assume that the magnetic field, which we will take to be in the z-direction. Write the vector potential in the form $\vec{A} = \hat{y}B(t)x$ so that the Hamiltonian for this system is given by $H(x,y,z,p_x,p_y,p_z) = \frac{p_x^2 + \left(p_y - qB(t)x\right)^2 + p_z^2}{2m}$. Thus p_y and p_z are conserved.

   a. Find v_x, v_y and v_z in terms of p_x, p_y, p_z, x, q, m and $B(t)$.

b. Since p_y and p_z are conserved one can treat them as constants and consider a Hamiltonian in terms of the x degrees of freedom only: $H(x,p_x) = \dfrac{p_x^2 + \left(p_y - qB(t)x\right)^2 + p_z^2}{2m}$. Show that this can be written in terms of action angle variables: $H(\omega,J) = \dfrac{p_z^2}{2m} + \dfrac{JqB(t)}{2\pi\ m}$ where

$$J = qB(t)\pi\left(x - \frac{p_y}{qB(t)}\right)^2 + \frac{\pi\, p_x^2}{qB(t)} \text{ and } \omega = \frac{1}{2\pi}\tan^{-1}\left(\frac{qB(t)\left(x - \frac{p_y}{qB(t)}\right)}{p_x}\right).$$

c. Show that where $\dfrac{v^2 - v_z^2}{B}$ (where v is the speed of the particle) is an adiabatic invariant.

d. Note that in the adiabatic limit any given time the particle is making circular orbits (with the radius varying in time as the magnetic field strength changes) show that the magnetic flux threading through the circular orbit is an adiabatic invariant.

e. Suppose a particle makes a circular orbit with radius r_o in a magnetic field B_o. What is the radius if the magnetic field is slow increased to 2 B_o.