Physics 601 Homework 10---Due Friday November 19

Goldstein 5.8, 5.14

1. Consider a rigid body with the following mass density:
   $\rho(\vec{r}) = \rho_0 \exp\left(-\frac{x^2 + y^2 + z^2 + xy}{2l^2}\right)$
   a. Find the moment of inertia tensor.
   b. Find the three principal moments of inertia.

2. Consider a rigid body with no external torques. Use Euler's equations to show that the energy associated with rotational motion given in the body-fixed frame by $E = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3}$ is conserved.

3. In discussing rotations one can associate the angle velocity vector $\vec{\omega}$ a tensor given by $\overleftrightarrow{\Omega} = -\vec{\omega} \cdot \overleftrightarrow{\vec{M}}$.
   a. Starting from the properties of the generator matrices show that $\omega_i = \frac{1}{2}\varepsilon_{ijk}\Omega_{jk}$.

   We know that $\vec{\omega}$ is a vector (i.e. that is that it transforms like a vector under rotations). The purpose of the remainder of this problem is to use this to demonstrate that $\overleftrightarrow{\Omega}$ is constructed to be a tensor.
   b. As a first step you will need to demonstrate that the Levi-Civita symbol ε_{ijk} transforms like a rank-three tensor under rotations: $\varepsilon'_{ijk} = R_{il}R_{jm}R_{kn}\varepsilon_{lmn}$ with ε' having the same form as ε (where the result for any rotation). (This relatively easy once you realize that any rotation may be represented as three rotations about fixed axes using the Euler angle construction.)
   c. Using the result in b. show that $\overleftrightarrow{\Omega}$ transforms like a tensor.

4. Consider a rigid symmetrical object with two of the principal moments of inertia equal $I_1 = I_2$ (and the third, I_3 unequal) and no external torques. In this case one fully solve the Euler equations. Suppose that at t=0 the initial angular velocities are $\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}$ find $\omega_1, \omega_2, \omega_3$ for all times. Discuss what your solution tells you about the frequency of the wobble in a badly thrown football (or Frisbee).

5. One problem with solving the Euler equations is that it gives you components angular velocity in the body-fixed frame. You may want the rotation matrix (or the Euler angles) as a function of time. This problem discuss how to convert from one to another:

a. Show that given $\vec{\omega}^{(body)}$ as a function of time $\vec{R}$ is the solution to the following differential equation . $\vec{R}(t) = \vec{R}(0) - \int_0^t dt' \vec{\omega}^{(body)}(t') \cdot \vec{\vec{M}} \vec{R}(t')$

b. In general this is not trivial to evaluate. Show that the solution to this equation can be written in the following series form:
$\vec{R}(t) = \vec{R}_0 + \vec{R}_1(t) + \vec{R}_2(t) + \vec{R}_3(t) \ldots$ with $\vec{R}_0 = \vec{R}(0)$ and
$\vec{R}_{n+1}(t) = -\int_0^t dt' \, \vec{\omega}^{(body)}(t') \cdot \vec{\vec{M}} \vec{R}_n(t')$.

c. The result in b. can be written in a compact form if one introduces the notion of time-ordered product of two matrice swhich are functions of time:
$T[\vec{a}(t)\vec{b}(t')] \equiv \begin{cases} \vec{a}(t)\vec{b}(t') & \text{for } t > t' \\ \vec{b}(t')\vec{a}(t) & \text{for } t' > t \end{cases}$ and more generally if one has a product of n different matrices, the time–ordered product is simply the product reorder in order descending time show that
$\vec{R}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} T\left[\left(-\int_0^t dt' \left(\vec{\omega}^{(body)}(t') \cdot \vec{\vec{M}} \right) \right)^n \right] \vec{R}(0)$. In showing this is sufficient for the purpose of the problem to demonstrate that it holds for the first few terms (up to n=3). Note that the time ordering is non-trivial since the *t'* in the integral is a dummy variable and product of two integrals involves integration of two distinct dummy variable. From the form above it is common to rewrite this as a "time-order exponential"
$\vec{R}(t) = T\left[\exp\left(-\int_0^t dt' \left(\vec{\omega}^{(body)}(t') \cdot \vec{\vec{M}} \right) \right) \right] \vec{R}(0)$.