Physics 601 Homework 6---Due Friday October 15

1. In class starting with the action $S = \int d\tau(-m - \boldsymbol{S})$ we used covariance to show that for a particle moving in a Lorentz scalar field $\frac{d\left((m+\boldsymbol{S})u^\mu\right)}{d\tau} = \partial^\mu \boldsymbol{S}$ where $\partial^\mu \equiv g^{\mu\nu}\frac{\partial}{\partial x^\nu}$.
   a. Show that this can be rewritten in the form $(m+\boldsymbol{S})\frac{d(u^\mu)}{d\tau} = \partial^\mu \boldsymbol{S} - (\partial^\alpha \boldsymbol{S})u_\alpha u^\mu$
   b. Show that this equation of motion automatically satisfies the condition $\frac{d(u_\mu u^\mu)}{d\tau} = 0$. This indicates that imposition of covariance yielded a self-consistent result that respects the condition $u_\mu u^\mu = 1$.
   c. Show that in the non-relativistic limit where all of the velocities are much less than the speed of light and $\boldsymbol{S} << m$ the Lagrangian for the system reduces to $L = \frac{1}{2}m\dot{\vec{x}}^2 - \boldsymbol{S}$ plus an irrelevant constant and the equation of motion reduces to $m\ddot{\vec{x}} = -\vec{\nabla}\boldsymbol{S}$.

2. Start from the action $S = \int d\tau\left(-m + V^\mu u_\mu\right)$ where A_μ is a four vector field that depends on space-time. Show that the equation of motion is $\frac{d(mu_\mu)}{d\tau} = \left(\frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}\right)u^\nu$.

3. In electro-magnetism, one can write the scalar and vector potentials in a form that looks like a 4-vector: $A^\mu = \begin{pmatrix}\Phi \\ A_x \\ A_y \\ A_z\end{pmatrix}$. Because one can make arbitrary gauge transformations A^μ need not transform as a 4-vector.

   a. Show that a gauge transformation can be written in the form of the form $A_\mu \to A'_\mu = A_\mu + \frac{\partial G}{\partial x^\mu}$ where G is an arbitrary function of space-time which need not transform as a 4-scalar under Lorentz transformations .
   b. A sufficient condition to show that A'^μ transforms as a 4-vector is to show that $\frac{\partial A'^\mu}{\partial x^\mu} = 0$ with $|A^\mu| \to 0$ as $|\vec{x}| \to \infty$ Explain briefly why.
   c. Show that it is always possible to make a gauge transformation (i.e. to choose G) to ensure that A'^μ *does* transform as a 4-vector by picking

choosing Λ to satisfy the condition $\partial_\mu A^\mu = -\partial_\mu \partial^\mu G$. This is called the Lorentz gauge.

d. Show that the field-strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant. That is show that $F'_{\mu\nu} \equiv \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu}$ for any transformation of the form given in part a.

e. Show that $F_{\mu\nu}$ transforms under Lorentz transformations as a 4-tensor: $F_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}$ where Λ is the matrix which specifies the Lorentz transformation.

f. Using the result problem 1, plus the preceding parts of this problem show that the equation of motion for a particle of mass *m* and charge *q* moving in an electro magnetic field is given by $\frac{d(mu^\mu)}{d\tau} = qF^{\mu\nu}u_\nu$.

4. In the preceding section you showed that $\frac{d(mu^\mu)}{d\tau} = qF^{\mu\nu}u_\nu$. Staring with this equation of motion show that of necessity $\frac{d\left(u_\mu u^\mu\right)}{d\tau} = 0$. This indicates that equation of motion self-consistently respects the condition $u_\mu u^\mu = 1$.

5. Suppose that one has a charged particle with mass *m* and charge *q* interacting with external electromagnetic fields specified by the potentials $\Phi = 0,\ A_x = -E_0 t,\ A_y = A_z = 0$. Note that potentials are independent of special position.

   a. Verify that these potentials satisfy the Lorentz gauge condition $\partial_\mu A^\mu = 0$.

   b. Construct the field strength tensor $F_{\mu\nu}$.

   c. Suppose that the particle starts from rest at *t*=0, find the position of the particle as a function of time. (Hint, you may find your solution of problem 4.d of homework 5 to be useful.)

6. In class we found the Green's function for the harmonic oscillator. In this problem, I want you to find and use the analogous one for a damped oscillator. The damped driven oscillator satisfied the equation: $m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2 x = f(t)$ where β is a damping parameter. The solution is $x(t) = \int_{-\infty}^{\infty} dt' G(t,t') f(t')$ where the Green's function satisfies $(\partial_t^2 + 2\beta\partial_t + \omega_0^2)G(t,t') = \delta(t-t')$. A useful first step in constructing this is to exploit the known solution for steady state motion with a harmonic driving force: $(\partial_t^2 + 2\beta\partial_t + \omega_0^2)x(t) = f_0 e^{i\omega t}$ has a solution of the form of the form $x(t) = \frac{f_0 e^{i\Omega(t-t')}}{\omega_0^2 - \Omega^2 + 2i\beta\Omega}$. Thus $(\partial_t^2 + 2\beta\partial_t + \omega_0^2)\frac{e^{i\Omega(t-t')}}{\omega_0^2 - \Omega^2 + 2i\beta\Omega} = e^{i\Omega(t-t')}$. Let us now integrate both sides with respect to Ω and divide by 2π:

$$\frac{\int_{-\infty}^{\infty} d\Omega\,(\partial_t^2 + 2\beta\partial_t + \omega_0^2)\dfrac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2i\beta\omega}}{2\pi} = \frac{\int_{-\infty}^{\infty} d\omega\, e^{\,i\omega(t-t')}}{2\pi}$$. We know that $\frac{\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}}{2\pi} = \delta(t-t')$. Moreover on the left hand side we can pull $(\partial_t^2 + 2\beta\partial_t + \omega_0^2)$ out of the integral as it does not depend on ω. Thus

$$(\partial_t^2 + 2\beta\partial_t + \omega_0^2)\left(\frac{\int_{-\infty}^{\infty} d\omega\, \dfrac{e^{i\omega(t-t')}}{\omega_0^2 - \omega^2 + 2i\beta\omega}}{2\pi}\right) = \delta(t-t')$$ and the object in the parenthesis is a Green's function.

a. Evaluate the integral above using contour integration to find an explicit expression for $G(t,t')$. Note that the complex exponential implies that the ½ plane in which the contour is to be closed depends on the sign of *t-t'*.
b. Use this Green's function to find a solution of $m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2 x = f_0\, e^{-\Gamma t}\theta(t)$.
c. Consider the result in b. in the regime where Γ>>ω>>β. In that regime the system should look like an underdamped oscillator getting a delta-function-like impulse at t=0. Does it?