# Physics 601 Homework 5---Due Friday October 8

1. A standard result in undergraduate relativity is the velocity addition formula: $v = \frac{v_1 + v_2}{1 + v_1 v_2}$. This typically obtained by taking the product of two Lorentz transformations and the doing some algebra. A more straightforward way to obtain this use the four velocity: $u^\mu = \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$ $\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$. Suppose I start with a particle moving with velocity $\vec{v}_1$ with an associated four-velocity $u_1^{\ \mu} = \begin{pmatrix} \gamma_1 \\ \gamma_1 v_{1x} \\ \gamma_1 v_{1y} \\ \gamma_1 v_{1z} \end{pmatrix}$.

   Suppose one boosts to a new frame by running to the left (-*x* direction) with a velocity which corresponds to a Lorentz transformation $\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma_2 & v_2\gamma_2 & 0 & 0 \\ v_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   The four-velocity in the new frame is $u^\nu = \Lambda^\mu{}_\nu u_1^{\ \nu}$.

   a. From the transformed four velocity find $v_x, v_y, v_z$ in the new frame.

   b. For the case where $\vec{v}_1$ is entirely along the x direction, show that $v = \frac{v_1 + v_2}{1 + v_1 v_2}$.

2. Consider relativistic transformations restricted to one spatial direction. In that case, the velocity can be specified by a single number from -1 to 1 (in units with *c*=1). It is convenient to introduce the "rapidity" $\eta$ with the property that $v = \tanh(\eta)$. Note that while *v* is restricted from -1 to 1, $\eta$ goes from -∞ to ∞.

   a. Show that the 4-velocity is given by $u^\mu = \begin{pmatrix} \cosh(\eta) \\ \sinh(\eta) \\ 0 \\ 0 \end{pmatrix}$.

   b. Show that while velocities do not add linearly in relativity rapidities do. That is one obtains the relativistic velocity addition formula by taking $\eta = \eta_1 + \eta_2$. In a certain sense this turns out to be just the hyperbolic trig version of the angle addition formula for two-dimensional rotations.

3. Relativistic Kinematics: Consider the elastic scattering of two particles with masses $m_1$ and $m_2$. Suppose initially particle 1 is at rest and and particle 2 approaches it

with a velocity *v* in the positive z directions after the scattering particle 1 goes off making an angle $\theta_1$ with respect to the z axis. The purpose of this problem is to find expressions for the magnitude of the velocity of the two particles and the angle made by the second particles in terms of *v* and $\theta_1$. Do this by first going to the center of mass frame (with zero total momentum) where energy and momentum conservation tell us the scattering must be back to back and then boosting back to the lab frame.

4. A rocket fires a constant rate so that rocket's own rest frame it experiences an acceleration in the +x direction with a magnitude of *g* (i.e. an observer in the rocket ship would feel an artificial gravity pushing them in the negative x direction with a force of *mg*).
    a. Show that the equation of motion for the rocket is given by $\frac{du^\mu}{d\tau} = G^{\mu\nu} u_\nu \quad \text{with} \quad G^{\mu\nu} = \begin{pmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$. To do this you need to show that the acceleration in the commoving frame is correct and that the form correctly gives $\frac{d\left(u_\mu u^\mu\right)}{d\tau} = 0$ as required from the definition of $u^\mu$.
    b. Work in a frame for which the rocketship starts at rest at the orgin at t=0. Find an expression for $u^\mu(\tau)$.
    c. Use the result of b. to compute $x^\mu(\tau)$.
    d. Find x(t).