Physics 601 Homework 4---Due Friday October 1

Goldstein: 9.7, 9.25

In addition:

1. This problem uses the result in problem 2 of homework 3 in the context of a 6-dimension phase space associated with a single particle in 3 dimensions. The canonical variables are x, y, z, p_x, p_y, p_z. In this problem various transformations with clear physical meanings are proposed. State the physical meaning, show that they are canonical and find the generator of the transformation (as defined in in problem 2 of homework 3) and show it satisfies the relation $\frac{\partial \vec{\xi}(\vec{\eta};\varepsilon)}{\partial \varepsilon} = [\vec{\xi}, g]_{PB}$ given in problem 2 of homework 3. In all of these ε is a constant.

$X = x + \varepsilon$	$X = x\cos(\varepsilon) + y\sin(\varepsilon)$	$X = x$	$X = (1+\varepsilon)x$
$Y = y$	$Y = -x\sin(\varepsilon) + y\cos(\varepsilon)$	$Y = y$	$Y = (1+\varepsilon)y$
$Z = z$	$Z = z$	$Z = z$	$Z = (1+\varepsilon)z$
$P_x = p_x$	$P_x = p_x\cos(\varepsilon) + p_y\sin(\varepsilon)$	$P_x = p_x + \varepsilon$	$P_x = (1+\varepsilon)^{-1} p_x$
$P_y = p_y$	$P_y = -p_x\sin(\varepsilon) + p_y\cos(\varepsilon)$	$P_y = p_y$	$P_y = (1+\varepsilon)^{-1} p_y$
$P_z = p_z$	$P_z = p_z$	$P_z = p_z$	$P_z = (1+\varepsilon)^{-1} p_z$
$a.$	$b.$	$c.$	d

2. Show that if a one-parameter family of time-independent canonical transformations leaves the Hamiltonian invariant and its generator has no explicit time dependence then its generator is conserved. That is given a canonical transformation $\vec{\xi}(\vec{\eta};\varepsilon)$ with $H(\vec{\xi}(\vec{\eta};\varepsilon)) = H(\vec{\eta})$ and with g as a generator---$\frac{\partial \vec{\xi}(\vec{\eta};\varepsilon)}{\partial \varepsilon} = [\vec{\xi}, g]_{PB}$----then g is conserved.

3. In problem 1 of HW 2 we showed an explicit example of a conserved quantity which was not associated with a point transformation. Use the result shown above in problem 2 to show that the conserved quantity Δ is associated with a family of canonical transformations by explicitly constructing the family of transformations.

4. Consider the canonical transformation for a system with one degree of freedom generated by: $F^2(q,P,t) = (q + \frac{1}{2}gt^2)(P - mgt) - \dfrac{P^2 t}{2m}$.
   a. Find the explicit form for the canonical transform.
   b. Verify that it satisfies the canonical Poisson bracket relations.

   For the remainder of this problem consider particle moving in one dimension in a constant gravitational field $L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - mgq$.

   c. Use the fact that for any A, $\dfrac{dA}{dt} = [A,H]_{PB} + \dfrac{\partial A}{\partial t}$ to show that $\dfrac{dQ}{dt} = 0$ and $\dfrac{dP}{dt} = 0$.
   d. Part c. implies that K the Hamiltonian associated with Q, P must be zero up to a possibly time dependent constant independent of Q, P. Show from the explicit form of $\dfrac{\partial F^2(q,P,t)}{\partial t}$ that this is in fact the case.
   e. What is the physical interpretation of Q, P.
   f. Show that $F^2(q,P,t)$ satisfies a Hamilton-Jacobi equation: $H\left(q, \dfrac{\partial F^2}{\partial q}\right) + \dfrac{\partial F^2}{\partial t} = 0$.
   g. Introduce the function $f(Q,P,t) = F^2(q(Q,P,t),P,t)$. Show that $\dfrac{\partial f(Q,P,t)}{\partial t} = L\big(q(Q,P,t), \dot{q}(Q,P,t)\big)$.

5. In class we derived the Hamilton Jacobi equation $H\left(\vec{q}, \vec{\nabla} S(\vec{q},\vec{P})\right) + \dfrac{\partial S(\vec{q},\vec{P})}{\partial t} = 0$ where $\vec{P}$ is a constant of the motion. Derive an analogus expression $\vec{Q}$ $H\left(-\vec{\nabla}_p \tilde{S}(\vec{Q},\vec{p}), \vec{p}\right) + \dfrac{\partial \tilde{S}(\vec{Q},\vec{p})}{\partial t} = 0$ where $\vec{Q}$ is a constant of the motion and the tilde is on $\tilde{S}$ to distinguish it from S.