

## Physics 601 Homework 2---Due Friday, September 17

Hint: For some of these problems it will be helpful to use Mathematica or some other symbolic manipulation program. If you make use of such a program please include the output with your homework solutions.

Goldstein 1.22, 2.20, 13.4

In addition:

1. In class we showed that for every invariance of a Lagrangian under a continuous point transformation there is an associated conserved quantity. The converse, however, is not true---every conserved quantity does not have an associated invariance under a point transformation. For example, we know that the energy is conserved (for systems with no explicit time dependence) even though it is not associated with invariance under a point transformation. One might consider the possibility that *apart* from energy all conserved quantities are necessarily associated with invariance of a Lagrangian under continuous point transformations. However this too is untrue as will be illustrated in this problem. Consider a general two-dimensional harmonic oscillator:  $L(x, y; \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + \alpha y^2)$  where  $\alpha$  is a parameter specifying the degree of anisotropy.
  - a. Show that the energy,  $E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\omega^2(x^2 + \alpha y^2)$ , is conserved.
  - b. Use the equations of motion to show that there is another conserved quantity  $\Delta \equiv \frac{1}{2}m(\dot{x}^2 - \dot{y}^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2)$ .
  - c. Show that any conserved current obtainable from a point transformation,  $\Gamma \equiv \left. \frac{\partial L}{\partial \dot{x}} \frac{\partial Q_1}{\partial \epsilon} \right|_{\epsilon=0} + \left. \frac{\partial L}{\partial \dot{y}} \frac{\partial Q_2}{\partial \epsilon} \right|_{\epsilon=0}$  with  $Q_1(x, y; \epsilon), Q_2(x, y; \epsilon)$  where  $Q_1(x, y; 0) = x, Q_2(x, y; 0) = y$ , is necessarily linear in the velocities  $\dot{x}$  and  $\dot{y}$ .
  - d. Explain why  $\Delta$  cannot be of the form of  $\Gamma$ .
  - e. Show that there is a conserved quantity of the form of  $\Gamma$  for a special value of  $\alpha$  but that it cannot be written as a linear combination of  $E$  and  $\Delta$ .
  
2. A principle advantage of using Lagrangians in describing continuum systems (field theories) is that is straightforward to include relativity since space and time are treated in an analogous way. This will be illustrated in the following simple problem in 1 space dimension. Lorentz transformations are given by
 
$$x' = \gamma x - \beta \gamma (ct) \quad ct' = \gamma (ct) - \beta \gamma x \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$
 where  $\beta = \frac{v}{c}$  is a parameter specifying the boost. The field under consideration is a so-called Lorentz scalar field  $\phi(x, t)$  with the

property that under Lorentz transformations  $\phi(x,t) \rightarrow \phi(x',t')$ . The Lagrangian density for the wave equation is given by

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}c^2(\partial_x \phi)^2$$

- a. Use the Euler-Lagrange equation for the action  $S = \int dx dt \mathcal{L}$  to show that the equation of motion for this system is the relativistic wave equation  $(\partial_t^2 - c^2 \partial_x^2)\phi(x,t) = 0$
  - b. Verify that the Lagrangian density,  $\mathcal{L}$ , is Lorentz invariant *i.e.* that the form of the transformed  $\mathcal{L}$  is identical to the untransformed one.
  - c. Verify that the action,  $S = \int dx dt \mathcal{L}$ , is Lorentz invariant. To do this one needs to find the Jacobian of the Lorentz transformation.
  - d. From part c. one would conclude that the form of the Euler Lagrangian equation is the same in all frames. Verify that this is true.
3. In class we showed that for particles in a magnetic field time-independent gauge transformations changed the action for motion between fixed points but did so in a manner independent of the path. In this problem you should show it is also the case for particles in an electro-magnetic field with time-dependent gauge transformations. The Lagrangian is  $L = \frac{1}{2} \dot{\vec{x}}^2 - q(\phi(\vec{x},t) + \vec{A}(\vec{x},t) \cdot \dot{\vec{x}})$  and the gauge transformation is given by
- $$\vec{A}(\vec{x},t) \rightarrow \vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) + \vec{\nabla} \Lambda(\vec{x},t) \quad \phi(\vec{x},t) \rightarrow \phi'(\vec{x},t) = \phi(\vec{x},t) - \frac{\partial \Lambda(\vec{x},t)}{\partial t}.$$