HOMEWORK # 11

1) $E = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3} = \sum_{i=1}^{3} \frac{J_i^2}{2I_i}$, $J_i = I_i \omega_i$

$$\dot{E} = \sum_{i=1}^{3} \underbrace{\frac{J_i}{I_i}}_{\omega_i} \dot{J}_i$$

For no torque Euler-equations become $\dot{J}_i = I_i \dot{\omega}_i = \epsilon_{ijk} J_j \omega_k = I_j \epsilon_{ijk} \omega_j \omega_k$

$$\dot{E} = \sum_{i=1}^{3} \underbrace{\omega_i \omega_j \omega_k I_j}_{} \epsilon_{ijk}$$

Symmetric in indices for $i \neq j \neq k$ for which $\epsilon_{ijk} \neq 0$ ↳ antisymmetric

$$\dot{E} = 0$$

(or by noting that $\dot{E} = \vec{\omega} \cdot \dot{\vec{J}}$ and $\dot{\vec{J}} \propto \vec{\omega} \times \vec{J}$ which is $\perp$ to $\vec{\omega}$. As a result $\dot{E} = \vec{\omega} \cdot (\vec{\omega} \times \vec{J}) = 0$)

2) a) Recall that rotations can be described as either active or passive.

A passive rotation (corresponding to an active one) rotates the basis vectors in the opposite direction (of the active rotation of vectors). Hence if $\vec{\omega}' = R\vec{\omega}$ then $\hat{e}_i' = R^T \hat{e}_i$.

$$\varepsilon_{ijk} = (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k$$

$$\varepsilon'_{ijk} = (\hat{e}_i' \times \hat{e}_j') \cdot \hat{e}_k' = (R^T \hat{e}_i \times R^T \hat{e}_j) \cdot R^T \hat{e}_k$$

R written in the basis of $\hat{e}_i$ gives $(R^T \hat{e}_i)_m = R^T_{mn} \overset{\delta_{in}}{(\hat{e}_i)_n} = R^T_{mi} = R_{im}$; $(R^T \hat{e}_j)_n = R_{jn}$

$$\varepsilon'_{ijk} = \underbrace{\epsilon_{mnl} (\hat{e}_i')_m (\hat{e}_j')_n}_{(\hat{e}_i' \times \hat{e}_j')_l} \underbrace{(\hat{e}_k')_l}_{(\hat{e}_k')_l} = \epsilon_{mnl} R_{im} R_{jn} R_{kl} \quad //$$

b) $\omega_i = \varepsilon_{ijk} \Omega_{jk}$ (We know Ω is a tensor)

$$\omega_i' = \varepsilon'_{ijk} \Omega'_{jk} = R_{im} R_{jn} R_{kl} \varepsilon_{mnl} \overbrace{\underbrace{R_{ja}}_{R^T_{aj}} \underbrace{R_{kb}}_{R^T_{bk}} \Omega_{ab}}$$

$$= R_{im} \underbrace{\underbrace{R^T_{aj} R_{jn}}_{(R^T R)_{an}}}_{\mathbb{1}_{an} = \delta_{an}} \underbrace{\underbrace{R^T_{bk} R_{kl}}_{(R^T R)_{bl}}}_{\mathbb{1}_{bl} = \delta_{bl}} \varepsilon_{mnl} \Omega_{ab} = R_{im} \overbrace{\varepsilon_{mnl} \Omega_{nl}}^{\omega_m} = R_{im} \omega_m \quad //$$

Let $I_3 = I$, $(I_2 - I_3) = (I_1 - I_3) = \Delta I$

3) $I_3 \dot{\omega}_3 = 0 \implies \omega_3 = \text{constant}$

$$\left.\begin{aligned} I_2 \dot{\omega}_2 &= -\Delta I \, \omega_1 \omega_3 \\ I_1 \dot{\omega}_1 &= \Delta I \, \omega_3 \omega_2 \end{aligned}\right\}$$

Solve by substituting one into the other

$$I_1 \ddot{\omega}_1 = \Delta I \omega_3 \dot{\omega}_2 = \Delta I \omega_3 \left(-\frac{\Delta I}{I_2} \omega_3\right) \omega_1$$

$$I_2 \ddot{\omega}_2 = -\Delta I \omega_3 \dot{\omega}_1 = -\Delta I \omega_3 \left(\frac{+\Delta I}{I_1} \omega_3\right) \omega_2$$

Letting $I_1 = I_2$

$$\implies \ddot{\omega}_1 + \left(\frac{\Delta I}{I_2} \omega_3\right)^2 \omega_1 = 0$$

$$\ddot{\omega}_2 + \left(\frac{\Delta I}{I_2} \omega_3\right)^2 \omega_2 = 0$$

Free harmonic oscillators with the same frequency $\Omega = \frac{|\Delta I|}{I_2} \omega_3$.

$$\omega_2(t) = \omega_2(0) \cos(\Omega t) + \frac{\dot{\omega}_2(0)}{\Omega} \sin(\Omega t) = \boxed{\omega_2(0)\cos(\Omega t) - \overbrace{\frac{\Delta I \, \omega_3 \omega_1(0)}{I_2 \Omega}}^{-\text{sign}(\Delta I)\, \omega_1(0)} \sin(\Omega t) = \omega_2(t)}$$

$$\omega_1(t) = \omega_1(0) \cos(\Omega t) + \frac{\dot{\omega}_1(0)}{\Omega} \sin(\Omega t) = \boxed{\omega_1(0)\cos(\Omega t) + \underbrace{\frac{\Delta I \, \omega_3 \omega_2(0)}{I_2 \Omega}}_{+\text{sign}(\Delta I)\, \omega_2(0)} \sin(\Omega t) = \omega_1(t)}$$

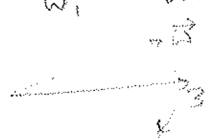
$$\boxed{\omega_3 = \omega_3(t)}$$

$$\dot{\omega}_2(0) = -\frac{\Delta I}{I_2} \omega_3 \omega_1(0)$$

$$\dot{\omega}_1(0) = \frac{\Delta I}{I_2} \omega_3 \omega_2(0)$$

For a "real" football (aka soccer ball) $I_1 = I_2 = I_3 \implies \Delta I = 0$, hence there is no wobble. For an "American" football  $I_2 = I_1 > I_3$. The frequency of wobble is $\Omega = \frac{|\Delta I|}{I_2} \omega_3$.

Also note that $\frac{\dot{\omega}_2}{\dot{\omega}_1} = -\frac{\omega_1}{\omega_2} \to \omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 = \frac{1}{2}\frac{d}{dt}(\omega_1^2 + \omega_2^2) = 0$, also $\omega_3 = \text{constant}$.

$\implies |\vec{\omega}|^2 = \text{constant}$. 

↳ a circle

4 a) Start with the definition $\vec{\Omega} = \vec{R}^T \dot{\vec{R}}$. Multiply by $\vec{R}$ from left and use $R R^T = \mathbb{1}$, to get $\dot{\vec{R}} = \vec{R}\vec{\Omega}$. Using $\vec{\Omega}^b = \vec{\omega}^b \cdot \vec{\vec{T}}$ we get

$\dot{\vec{R}} = \vec{R}\,\vec{\omega}^b\cdot\vec{\vec{T}}$. Integrate using initial condition $\vec{R}(t=0) = \vec{R}_0$

$$\vec{R}(t) = \vec{R}_0 + \int_0^t \vec{R}(t')\,\vec{\omega}^b\cdot\vec{\vec{T}}\,dt' \quad //$$

(I will drop the arrows on R)

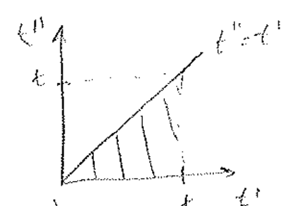
b) Substitute $R = R_0 + R_1(t) + R_2(t) + \dots$

$$\text{RHS} = R_0 + \int_0^t \left(R_0 + R_1(t') + R_2(t') + \dots\right)\vec{\omega}^b\cdot\vec{\vec{T}}\,dt'$$

$$\text{LHS} = R_0 + R_1(t) + R_2(t) + \dots$$

$$\Rightarrow \quad R_n(t) = \int_0^t R_{n-1}(t')\,\vec{\omega}^b\cdot\vec{\vec{T}}\,dt' \quad //$$

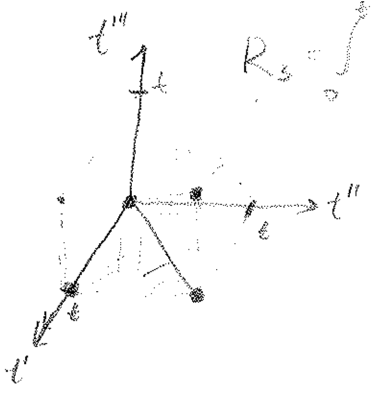
c)
$$R_0 = R_0$$
$$R_1 = \int_0^t R_0\,\vec{\omega}^b(t')\cdot\vec{\vec{T}}\,dt'$$
$$R_2 = \int_0^t R_1(t')\,\vec{\omega}^b(t')\cdot\vec{\vec{T}}\,dt' = \int_0^t \underbrace{\int_0^{t'} R_0\,\vec{\omega}^b(t'')\cdot\vec{\vec{T}}\,dt''}_{R_1(t')}\,\vec{\omega}^b(t')\cdot\vec{\vec{T}}$$



The integration in the t'-t'' plane is performed over the area. By the symmetry of the integrand integration over the entire square would be just twice that of the triangle. Time ordering keeps (preserves) the order of operators in the unshaded region.

$$\Rightarrow \int_0^t\int_0^{t'} = \frac{1}{2}T\int_0^t\int_0^t = \frac{1}{2}T\left(\int_0^t\right)\left(\int_0^t\right) = \frac{1}{2}T\left(\int_0^t\right)^2$$

$$R_3 = \int_0^t R_2(t')\,\vec{\omega}(t')\cdot\vec{\vec{T}}\,dt' = \int_0^t\int_0^{t'}\int_0^{t''} R_0\, w(t''')\cdot T\, w(t'')\cdot T\, w(t')\cdot T\,dt'\,dt''\,dt'''$$



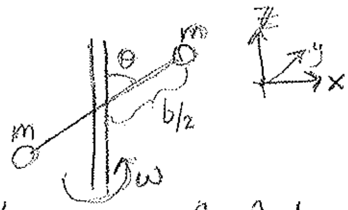
The integration volume is this time the tetrahedron with corners located at $(0,0,0)$, $(t,0,0)$, $(t,t,0)$, (t,t,t). The volume of this tetrahedron is $\frac{1}{6}t^3$, $\frac{1}{6}$th of that of the cube. Again using the symmetry of the integrand we can extend the integration to all volume, and argue like above. Time ordering is necessary to protect the order in which operators appear in integrand.

~~[illegible]~~
Phys 601

HW 10
Due 11/20

Jose 8.19

a)



$$R = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

First we calculate I_{body}, since that's easiest.

$I_{ij} = m(r^2\delta_{ij} - x_i x_j)$ for one mass

- $I_{xx} = m(r^2 - x^2) = mz^2 = m\frac{b^2}{4}\cos^2\theta$
- $I_{xy} = m(-xy) = 0$
- $I_{xz} = m(-xz) = -m\frac{b^2}{4}\sin\theta\cos\theta$
- $I_{yx} = I_{xy}$ $\quad I_{yy} = mr^2 = m\frac{b^2}{4}$
- $I_{yz} = m(-zy) = 0$
- $I_{zx} = I_{xz}$
- $I_{zy} = I_{yz}$
- $I_{zz} = m(r^2 - z^2) = mx^2 = m\frac{b^2}{4}\sin^2\theta$

The I's are the same for each mass, so we double the above to get

$$I_b = \frac{mb^2}{2}\begin{pmatrix} \cos^2\theta & 0 & -\sin\theta\cos\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta & 0 & \sin^2\theta \end{pmatrix}$$

Now, $I_S = R I_b R^T$, so using I_b and R from above,

$$I_S = \frac{mb^2}{2}\begin{pmatrix} \cos^2\theta\cos^2\omega t + \sin^2\omega t & -\sin^2\theta\sin\omega t\cos\omega t & -\sin\theta\cos\theta\cos\omega t \\ -\sin^2\theta\sin\omega t\cos\omega t & \cos^2\theta\sin^2\omega t + \cos^2\omega t & -\sin\theta\cos\theta\sin\omega t \\ -\sin\theta\cos\theta\cos\omega t & -\sin\theta\cos\theta\sin\omega t & \sin^2\theta \end{pmatrix}$$

We know $\Omega^S = \dot{R}R^T$ so

$$\Omega^S = \omega\begin{pmatrix} -\sin\omega t & -\cos\omega t & 0 \\ \cos\omega t & -\sin\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \omega\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Additionally, $\Omega^b = R^T\Omega^S R$, which yields the same result.

We now want to explicitly show $r \times p = J$.

$$r = \begin{pmatrix} \frac{b}{2}\sin\theta \\ 0 \\ \frac{b}{2}\cos\theta \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} 0 \\ 2m\frac{b}{2}\omega\sin\theta \\ 0 \end{pmatrix}$$

but these are not rotated →

We really want $r(t)$, $p(t)$. So we act on each w/ R:

$$Rr = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{b}{2}\sin\theta \\ 0 \\ \frac{b}{2}\cos\theta \end{pmatrix} = \frac{b}{2}\begin{pmatrix} \sin\theta\cos\omega t \\ \sin\theta\sin\omega t \\ \cos\theta \end{pmatrix}$$ and

$$Rp = m\omega b\begin{pmatrix} -\sin\omega t\sin\theta \\ \cos\omega t\sin\theta \\ 0 \end{pmatrix}.$$ Now we calculate the cross product.

$$r\times p = \frac{m\omega b^2}{2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\omega t & \sin\theta\sin\omega t & \cos\theta \\ -\sin\theta\sin\omega t & \sin\theta\cos\omega t & 0 \end{vmatrix} = \frac{m\omega b^2}{2}\begin{pmatrix} -\cos\theta\sin\theta\cos\omega t \\ -\cos\theta\sin\theta\sin\omega t \\ \sin^2\theta \end{pmatrix}$$

This is indeed $I_S \cdot \vec{\omega}$, so $I\omega = J$.

b) $$T = \dot{J} = \frac{1}{2}m\omega^2 b^2 \begin{pmatrix} \sin\theta\cos\theta\sin\omega t \\ -\sin\theta\cos\theta\cos\omega t \\ 0 \end{pmatrix}$$ This torque is exerted by the central shaft, which is rigid. (An elastic shaft would bend during the motion)

(or using $N = \omega \wedge (I\vec{\omega}) + \cancel{I\dot{\vec{\omega}}}^{\,0}$ directly in S)

c) We can think of the motion of this bar as that of the badly thrown football (American) of problem 3. Here we have 2 principle axes with same principle moments and the third one different. Also initially the motion is not entirely in one eigendirection, unless $\theta = 0$ or $\theta = 90°$. So please refer to problem 3 in this set for details.